Complex Analysis

Oliver Lorscheid

Lecture notes,
Groningen, 2023/24

These lecture notes were written in the fall semester of 2023/24, with the help of Jamara Admiraal, Prakhar Agarwal, Margherita Bonan, Manuel Forster, Anna Van Gaal, Sebastian Holwitt, Aiva Misevičiūtė Albert Silvans, Lars Wieringa, and Thomas Zwartveld. Special thanks go to Anna and Thomas for organizing the proof-reading and to Aiva and Margherita for the beautiful illustrations.

Contents

Introduction

Complex Analysis is the theory of complex differentiable functions from regions of the complex plane to the complex plane itself. In contrast to the formal similarity with real differentiable functions, it turns out that complex differentiability is a stronger condition that leads to a theory of extreme elegance and beauty.

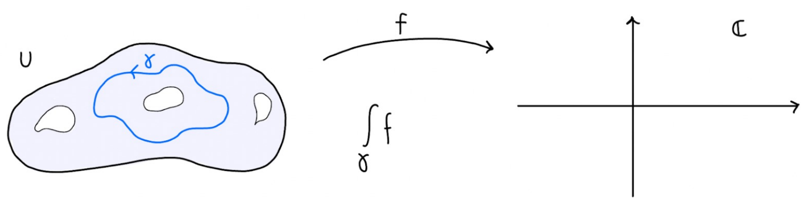
Let us compare the world of real valued functions $f : U \to \mathbb{R}$ and complex valued functions $f : U \to \mathbb{C}$ (where U is an open subset of $\mathbb{R}$ in the former case and of $\mathbb{C}$ in the latter case) in the following table.

real valued functions	complex valued functions
continuous	continuous
differentiable	holomorphic (complex diff'ble)
C^1 (cont. diff'ble)	$\vdots$
C^2	$\vdots$
$\vdots$	$\vdots$
C^∞ (smooth)	$\vdots$
analytic	analytic

While all classes of real valued functions (on the left hand side of the table) are properly contained in each other, holomorphic functions fulfill the following central result (which appears as Theorem 3.4.4 in the main text).

Theorem. *Every holomorphic function is analytic.*

The main technique of proof are path integrals of holomorphic functions:



Two corner stones towards the proof of the above theorem are the following (cf. Theorem 2.7.4 and Theorem 3.1.1 for the concise formulations).

Theorem (Cauchy's integral theorem)**.** *The path integral $\int_\gamma f$ is homotopy invariant.*

Theorem (Cauchy's integral formula)**.** *For a circular path γ around z (as below),*

$$f(z) = \frac{1}{2\pi \mathbf{i}} \cdot \int_\gamma \frac{f(w)}{w-z}\, dw.$$

These results pave the way to the identification of a path integral with an elementary expression in terms of easily computable numbers: the winding number $W(\gamma, c)$ of a closed path around a singularity c and the residue $\operatorname{Res}_c(f)$ of f at a singularity c (cf. Theorem 4.4.4 for a concise formulation).

Theorem (Cauchy's residue theorem)**.** *Let f be a holomorphic function and S the set of isolated singularities of f. Then*

$$\int_\gamma f = 2\pi \mathbf{i} \cdot \sum_{c \in S} W(\gamma, c) \cdot \operatorname{Res}_c(f).$$

Besides the aforementioned concepts and techniques, we learn in this course about:

- the Cauchy-Riemann equations (and the relation between holomorphic and harmonic functions);
- analytic extensions of the exponential function and trigonometric functions; branches of the logarithm;
- Taylor and Laurent expansions;
- types of singularities: removable singularities, poles and essential singularities.

We apply the methods and results of the course to establish the following results:

- Liouville's theorem;
- the fundamental theorem of algebra;
- the mean value theorem and the maximum modulus principle;
- Rouche's theorem;
- the open mapping principle;
- the inverse function theorem.

Chapter 1

Preliminaries

In this chapter, we revise knowledge from previous courses that we need for this course in Complex Analysis.

1.1 The complex numbers

Let $\mathbb{R}$ be the real numbers. The *complex numbers* are the set $\mathbb{C} = \mathbb{R}^2$ together with the vector addition

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x+x' \\ y+y' \end{pmatrix}$$

and the multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xx'-yy' \\ xy'+x'y \end{pmatrix}$$

which turn $\mathbb{C}$ into a field. In particular, the additive neutral element of $\mathbb{C}$ is $\binom{0}{0}$ and the multiplicative unit is $\binom{1}{0}$. The additive inverse of $\binom{x}{y}$ is $-\binom{x}{y} = \binom{-x}{-y}$ and the multiplicative inverse of a nonzero element $\binom{x}{y}$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}^{-1} = \frac{1}{x^2+y^2} \cdot \begin{pmatrix} x \\ -y \end{pmatrix}.$$

We denote by $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ the *unit group* of $\mathbb{C}$.

The *real part* of a complex number $\binom{x}{y}$ is $\operatorname{Re}\binom{x}{y} = x$ and its imaginary part is $\operatorname{Im}\binom{x}{y} = y$. We call $\{\binom{x}{0} \mid x \in \mathbb{R}\}$ the *real axis* of $\mathbb{C}$ and $\{\binom{0}{y} \mid y \in \mathbb{R}\}$ its *imaginary axis*. Note that the real axis forms a subfield of $\mathbb{C}$ that is isomorphic to $\mathbb{R}$. In the following, we identify $\mathbb{R}$ with the real axis and write x for $\binom{x}{0}$.

The *imaginary unit* is the element $\mathbf{i} = \binom{0}{1}$, which satisfies $\mathbf{i}^2 = \binom{-1}{0} = -1$, i.e. $\mathbf{i}$ is a square root of -1. This allows us to rewrite a complex number $\binom{x}{y}$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \mathbf{i} \cdot \begin{pmatrix} y \\ 0 \end{pmatrix} = x + \mathbf{i}y.$$

In this notation, the previous formulas for the sum, product and inverse of complex numbers $z = x + \mathbf{i}y$ and $z' = x' + \mathbf{i}y'$ can be rewritten as

$$\begin{aligned} z + z' &= (x + x') + \mathbf{i}(y + y'), \\ z \cdot z' &= (xx' - yy') + \mathbf{i}(xy' + x'y), \\ z^{-1} &= \frac{x - \mathbf{i}y}{x^2 + y^2}. \end{aligned}$$

The *complex conjugate* of $z = x + \mathbf{i}y$ is $\bar{z} = x - \mathbf{i}y$. The *absolute value* of $z = x + \mathbf{i}y$ is

$$|z| \ = \ \sqrt{z \cdot \bar{z}} \ = \ \sqrt{x^2 + y^2}.$$

Thus we have $z^{-1} = \bar{z}/|z|^2$.

We typically illustrate $\mathbb{C} = \mathbb{R}^2$ as a plane with $\mathbf{1}$ pointing to the right and $\mathbf{i}$ pointing upwards. Complex conjugation corresponds to the reflection in the x-axis.

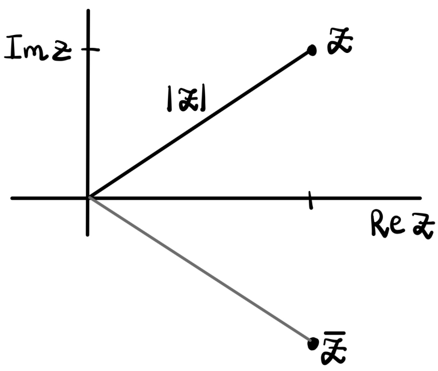


Figure 1.1: Complex conjugation as the reflection in the x-axis

1.2 Polar coordinates

Let $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ be the set of positive real numbers and $\mathbb{R}/2\pi\mathbb{Z}$ the quotient of the additive group of $\mathbb{R}$ by the subgroup $2\pi\mathbb{Z}$. The map

$$\begin{array}{rccc} \Phi : & \mathbb{R}_{>0} \times (\mathbb{R}/2\pi\mathbb{Z}) & \longrightarrow & \mathbb{C}^\times \\ & (r, \varphi) & \longmapsto & r(\cos\varphi + \mathbf{i}\sin\varphi) \end{array}$$

is a bijection whose inverse sends $z \in \mathbb{C}^\times$ to $\Psi(z) = (|z|, \arg z)$ where $\arg z$ is the *argument* of z;[1] this is, up to multiples of 360 degrees (i.e. 2π radians), the angle between the positive real axis and the halfline spanned by z in counterclockwise direction. The tuple $(|z|, \arg z)$ is called the *polar coordinates* of z.

[1]Note that we consider the argument as a function to $\mathbb{R}/2\pi\mathbb{Z}$, i.e. the argument is only well-defined up to integers multiples of 2π. This allows us to add angles without further explanations, in contrast to the so-called *principal argument*, which takes values in $(-\pi, \pi]$.

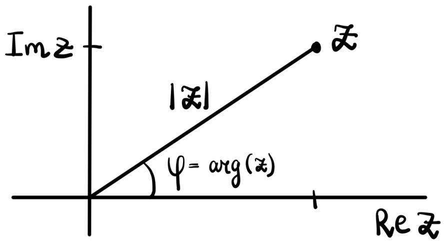


Figure 1.2: Argument of z

We can express several formulas in polar coordinates:

$$\begin{aligned} \Psi(z \cdot z') &= (|z| \cdot |z'|,\ \arg z + \arg z'), \\ \Psi(z^{-1}) &= (|z|^{-1}, -\arg z), \\ \Psi(\bar{z}) &= (|z|, -\arg z). \end{aligned}$$

In particular, the multiplication with a complex number z of absolute value $|z| = 1$ is a counter-clockwise rotation of $\mathbb{C} = \mathbb{R}^2$ by the angle $\frac{360}{2\pi} \cdot \arg z$ around 0.

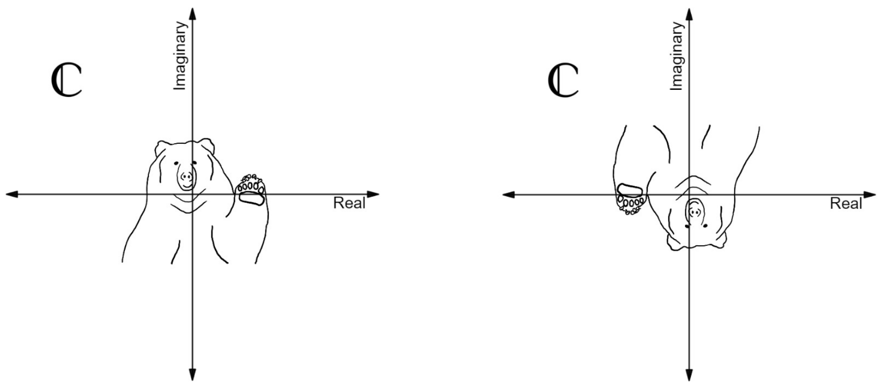


Figure 1.3: A picture of a bear and its image after multiplication by $z = -1$

1.3 Sequences and series

A *sequence in* $\mathbb{C}$ is an indexed family $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers $z_n \in \mathbb{C}$. A *limit* of a sequence $\{z_n\}_{n\in\mathbb{N}}$ is a complex number $z \in \mathbb{C}$ such that for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|z_k - z| < \epsilon$ for all $k > N$. A sequence $\{z_n\}_{n\in\mathbb{N}}$ *converges* if a limit exists. Otherwise it is said to *diverge*. We write

$$z = \lim_{n\to\infty} z_n \qquad \text{or} \qquad z_n \underset{n\to\infty}{\longrightarrow} z$$

to indicate that z is a limit of $\{z_n\}_{n\in\mathbb{N}}$. A *Cauchy sequence in* $\mathbb{C}$ is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $k, l \geqslant N$, we have $|z_k - z_l| < \epsilon$.

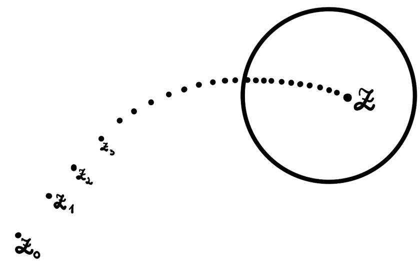


Figure 1.4: Convergence of a Cauchy sequence

Fact 1.3.1. *Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$.*

(1) Limits are unique, i.e. $\{z_n\}_{n\in\mathbb{N}}$ has at most one limit.

(2) The sequence $\{z_n\}_{n\in\mathbb{N}}$ has a limit if and only if it is a Cauchy sequence.

A *series in* $\mathbb{C}$ is an expression of the form $\sum_{i=0}^{\infty} z_n$ for complex numbers $z_n \in \mathbb{C}$. The *partial sums* of $\sum_{i=0}^{\infty} z_i$ are the sums $S_n = \sum_{i=0}^{n} z_i$. The series $\sum_{i=0}^{\infty} z_i$ *converges* (to z) if the sequence $\{S_n\}_{m\in\mathbb{N}}$ converges (to z). The series $\sum_{i=0}^{\infty} z_i$ *converges absolutely* if $\sum_{i=0}^{\infty} |z_i|$ converges as a series in $\mathbb{R}$.

Fact 1.3.2. *A series in $\mathbb{C}$ converges if it converges absolutely.*

Definition 1.3.3. Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers a_i. The *superior limit* of $\{a_i\}$ is

$$\limsup a_i \;=\; \lim_{n\to\infty} \big(\sup\{a_i \mid i \geqslant n\}\big)$$

which we interpret as ∞ if $\sup\{a_i \mid i \geqslant n\} = \infty$ for all n and as $-\infty$ if $\sup\{a_i \mid i \geqslant n\}$ has no lower bound.

Note that the superior limit always exists since $\sup\{a_i \mid i \geqslant n\}$ is a monotonically decreasing sequence in n. If $\{a_i\}$ converges, then $\limsup\{a_i\} = \lim\{a_i\}$.

Fact 1.3.4. *Let $\{a_i\}$ and $\{b_i\}$ be sequences with superior limits in $\mathbb{R}$. Then*

$$\limsup a_i \;\pm\; \limsup b_i \;=\; \limsup(a_i \pm b_i)$$

and

$$\limsup a_i \;\cdot\; \limsup b_i \;=\; \limsup(a_i \cdot b_i).$$

If $a_i \neq 0$ for all $i \in \mathbb{N}$ and $\limsup a_i \neq 0$, then

$$\limsup(a_i^{-1}) \;=\; (\limsup a_i)^{-1}.$$

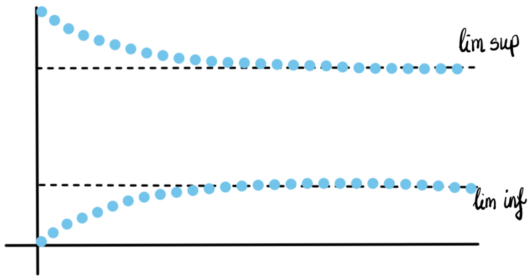


Figure 1.5: Limit superior and limit inferior

1.4 Open and closed subsets

We are mostly concerned with the topology of $\mathbb{C} = \mathbb{R}^2$, but at times, we also apply topological properties to $\mathbb{R} = \mathbb{R}^1$. Therefore we revise the following topological concepts and facts in the generality of $\mathbb{R}^n$ for $n \geqslant 0$.

The *Euclidean norm* of a vector $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ is

$$\|x\| \;=\; \sqrt{x_1^2 + \dots + x_n^2}.$$

For $x \in \mathbb{R} = \mathbb{R}^1$, the Euclidean norm is equal to the usual Euclidean absolute value $\|x\| = |x|$. For $z \in \mathbb{C} = \mathbb{R}^2$, the Euclidean norm is equal to the complex absolute value $\|z\| = |z|$.

Let $r > 0$ and $a \in \mathbb{R}^n$. The *open ball* of radius r and with center a is the set

$$B_r(a) \;=\; \{x \in \mathbb{R}^n \mid \|x - a\| < r\}.$$

In $\mathbb{R} = \mathbb{R}^1$, the open ball $B_r(a)$ is the open interval $(a - r,\ a + r)$. In $\mathbb{C} = \mathbb{R}^2$, the open ball $B_r(a)$ is the *open disc* $D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}$.

A subset U of $\mathbb{R}^n$ is *open* if for every $a \in U$, there is an $\epsilon > 0$ such that $B_\epsilon(a) \subset U$. In other words, U is open if and only if it is the union of open discs. A subset A of $\mathbb{R}^n$ is *closed* if $\mathbb{R}^n \setminus A$ is open.

Fact 1.4.1. *A subset A of $\mathbb{R}^n$ is closed if and only if every Cauchy sequence $\{z_n\}_{n \in \mathbb{N}}$ in A has a limit $z = \lim z_n$ in A.*

The *interior* of a subset W of $\mathbb{R}^n$ is

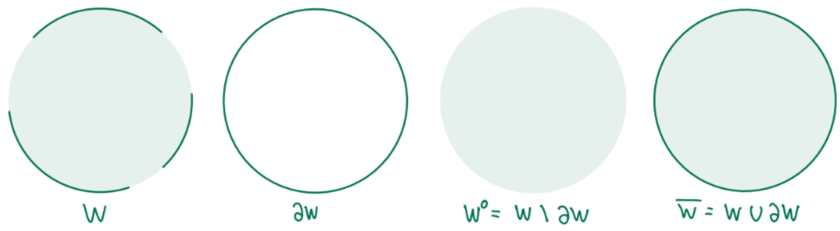
$$W^\circ \;=\; \bigcup_{\substack{U \subset W \\ \text{open}}} U \;=\; \{x \in W \mid B_\epsilon(w) \subset W \text{ for some } \epsilon > 0\},$$

which is the largest open subset of W. The *closure* of W is

$$\overline{W} \;=\; \bigcap_{\substack{W \subset A \\ \text{closed}}} A \;=\; \{w \in \mathbb{R}^n \mid w = \lim z_n \text{ for } \{z_n\}_{n \in \mathbb{N}} \subset W\},$$

which is the smallest closed subset of $\mathbb{R}^m$ that contains W. A subset $W \subset \mathbb{R}^n$ is *dense in* $\mathbb{R}^n$ if $\overline{W} = \mathbb{R}^n$. The *boundary* of W is the difference $\partial W = \overline{W} \setminus W^\circ$ between the closure and the interior of W. In particular, we have

$$W^\circ = W \setminus \partial W \qquad \text{and} \qquad \overline{W} = W \cup \partial W.$$



A subset V of W is *open in* W if it is of the form $V = U \cap W$ for an open subset U of $\mathbb{R}^n$. This topology for W is called the *induced topology* or the *subspace topology*.

1.5 Continuous functions

Let W be a subset of $\mathbb{R}^n$. A function $f : W \to \mathbb{R}^m$ has a *limit* $a \in \mathbb{R}^m$ in $w \in W$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $f(B_\delta(w) \cap W) \subset B_\epsilon(a)$. We write

$$a = \lim_{z \to w} f(z) \qquad \text{or} \qquad f(z) \underset{z \to w}{\longrightarrow} a$$

in this case. Note that the limit $\lim_{z\to w} f(z)$ of f in w is unique if it exists.

Fact 1.5.1. *Let $f, g : W \to \mathbb{R}^m$ be two functions for which the limit in $w \in W$ exists. Then*

$$\lim_{z\to w} (f \pm g)(z) = \lim_{z\to w} f(z) \pm \lim_{z\to w} g(z)$$

and

$$\lim_{z\to w} (f \cdot g)(z) = \lim_{z\to w} f(z) \cdot \lim_{z\to w} g(z).$$

If $\lim_{z\to w} f(z) \neq 0$, then

$$\lim_{z\to w} 1/f(z) = \left(\lim_{z\to w} f(z)\right)^{-1}.$$

The function $f : W \to \mathbb{R}^m$ is *continuous* if $f(w) = \lim_{z\to w} f(z)$ for every $w \in W$.

Fact 1.5.2. *Let $W \subset \mathbb{R}^n$ be a subset and $f : W \to \mathbb{R}^m$ a function. Then the following are equivalent:*

(1) The function f is continuous.

(2) For every open subset U in $\mathbb{R}^m$, the inverse image $f^{-1}(U)$ is open in W.

(3) For every sequence $\{z_n\}_{n\in\mathbb{N}}$ in W with limit $z \in W$, the image $f(z)$ is a limit of $\{f(z_n)\}_{n\in\mathbb{N}}$.

1.6 Connected subsets

A subset W of $\mathbb{R}^n$ is *connected* if it is not contained in the disjoint union $U_1 \sqcup U_2$ of two open subsets U_1 and U_2 of $\mathbb{R}^n$ that do neither contain W. A *path in* W is a continuous map $\gamma : I \to W$ from the unit interval $I = [0,1]$ to W. A subset W of $\mathbb{R}^n$ is *path connected* if for all $x, y \in W$, there is a path $\gamma : I \to W$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Fact 1.6.1. *Every path-connected subset W of $\mathbb{R}^n$ is connected. If W is open, then W is connected if and only if it is path-connected.*

1.7 Compact subsets

A subset W of $\mathbb{R}^n$ is bounded if there is an $r > 0$ such that $W \subset B_r(0)$. A subset A of $\mathbb{R}^n$ is *compact* if it is closed and bounded. For example, a subset of $\mathbb{R} = \mathbb{R}^1$ is compact if and only if it is a finite union of closed intervals. The *closed disc* $\overline{D}_r(a) = \{z \in \mathbb{C} \mid |z - a| \leqslant r\}$ of radius r and with center $a \in \mathbb{C}$ is a compact subset of $\mathbb{C}$.

Theorem 1.7.1 (Heine-Borel and Bolzano-Weierstrass)**.** *Let A be a subset of $\mathbb{R}^n$. The following are equivalent:*

(1) A is compact.

(2) If $A \subset \bigcup_{i \in I} U_i$ for open subsets U_i of $\mathbb{R}^n$, then there exists a finite subset J of I such that $A \subset \bigcup_{i \in J} U_i$.

(3) For every sequence $\{z_n\}_{n \in \mathbb{N}}$ in A, there is an order-preserving injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\{z_{\sigma(n)}\}_{n \in \mathbb{N}}$ has a limit in A.

Lemma 1.7.2. *Let $A \subset \mathbb{R}^n$ be a compact subset and $f : A \to \mathbb{R}^m$ a continuous function. Then $f(A)$ is a compact subset of $\mathbb{R}^m$. In particular, if $A \subset \mathbb{C}$ is compact, then every continuous function $f : A \to \mathbb{R}$ assumes its maximum, i.e. there is an $a \in A$ such that $f(z) \leqslant f(a)$ for all $z \in A$.*

Lemma 1.7.3 (Lebesgue's Lemma)**.** *Let $A \subset \mathbb{R}^n$ be a compact subset that is contained in the union $\bigcup_{i \in I} U_i$ of open subsets U_i of $\mathbb{R}^n$. Then there is a $\delta > 0$ such that for every $a \in A$, there is an $i \in I$ such that $B_\delta(a) \subset U_i$.*

In particular, if $[0,1]^n = \bigcup_{i \in I} U_i$ for open subsets U_i of $[0,1]^n$ (in the subset topology for $[0,1]^n$), then there is an $N > 0$ such that for all $k_1, \dots, k_n \in \{1, \dots, N\}$, there is an $i \in I$ such that

$$\left[\frac{k_1 - 1}{N}, \frac{k_1}{N}\right] \times \cdots \times \left[\frac{k_n - 1}{N}, \frac{k_n}{N}\right] \quad \subset \quad U_i.$$

1.8 Exercises

Exercise 1.8.1. Verify all formulas of section 0.1 and section 0.2.

Exercise 1.8.2. Show that $\lim_{n \to \infty} (n+1)^{1/n} = 1$, which can be done as follows:

(1) For an integer $n \geqslant 1$, show that $n+1 \leqslant \left(1+\sqrt{2/(n+1)}\,\right)^n$ by comparing $n+1$ with the binomial expansion of the right hand side.

(2) Use this together with $1 \leqslant (n+1)^{1/n}$ and $\lim 1+\sqrt{2/(n+1)} = 1$ to conclude that $\lim(n+1)^{1/n} = 1$.

Exercise 1.8.3. Let $z \in \mathbb{C}$ have $|z| < 1$. Show that the partial sum $S_n = \sum_{i=0}^{n} z^i$ equals $(z^{n+1}-1)/(z-1)$ and conclude that the series $\sum_{i=0}^{\infty} z^i$ converges absolutely to $(1-z)^{-1}$.

Exercise 1.8.4 (Warsow curve)**.** Consider the following subset of $\mathbb{R}^2$:

$$X \;=\; \{(x, \sin(\tfrac{1}{x}) \mid x > 0\} \;\cup\; \{(0,y) \mid -1 \leqslant y \leqslant 1\}.$$

Show that X is closed and connected. Show that X is not path-connected.

Exercise 1.8.5. Let $f : X \to Y$ be a continuous map, X compact and $Z \subset X$ closed. Prove that both Z and $f(X)$ are compact.

Exercise 1.8.6. Show that X is connected if and only if there is no continuous surjection $X \to \{0,1\}$ with respect to the discrete topology for $\{0,1\}$ (i.e. every subset is open).

Exercise 1.8.7. Prove all facts in this chapter.

Chapter 2

Holomorphic functions

2.1 Definitions

Definition 2.1.1. Let U and V be open subsets of $\mathbb{C}$ and $a \in U$. A function $f : U \to V$ is *complex differentiable in a* if

$$f'(a) \;=\; \lim_{z\to a} \frac{f(z) - f(a)}{z-a}$$

exists. A function $f : U \to V$ is *holomorphic* if it is complex differentiable in all $a \in U$. An *entire function* is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

Mostly the codomain V plays a subordinate role and it does not matter if we replace V by $\mathbb{C}$, or vice versa. Therefore we consider $f : U \to \mathbb{C}$ often as a function in $\mathbb{C}$. In some situations, it is however important to work with proper subsets V of $\mathbb{C}$ as a codomain, such as in Theorem 1.2.5.

Remark 2.1.2. A function $f : U \to \mathbb{C}$ is complex differentiable in $a \in U$ if and only if there is a $w \in \mathbb{C}$ such that

$$\frac{f(z) - f(a) - w\cdot(z-a)}{|z-a|} \;\underset{z\to a}{\longrightarrow}\; 0.$$

In this case $f'(a) = w$.

Note further that if $f'(a)$ exists, then

$$\lim_{z\to a}\big(f(z) - f(a)\big) \;=\; f'(a)\cdot \lim_{z\to a}(z-a) \;=\; 0,$$

which shows that f is continuous in a.

Proposition 2.1.3. *Let $U \subset \mathbb{C}$ be an open subset, $a \in U$, $c \in \mathbb{C}$ and $f,g : U \to \mathbb{C}$ two functions that are complex differentiable in a. Then*

(1) $(f \pm g)'(a) = f'(a) \pm g'(a)$;

(2) $(c \cdot f)'(a) = c \cdot f'(a)$;

(3) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$; *(Leibniz rule)*

(4) $(1/f)'(a) = -f'(a)/(f(a))^2$ *if* $f(a) \neq 0$.

(5) *If* $V \subset \mathbb{C}$ *is an open subset with* $f(U) \subseteq V$ *and* $h : V \to \mathbb{C}$ *is complex differentiable in* $f(a)$, *then* $(h \circ f)'(a) = h'(f(a)) \cdot f'(a)$. *(Chain rule)*

Proof. We only prove the most difficult case (5) and leave (1)–(4) as an exercise. The chain rule follows from the direct computation

$$\begin{aligned} \lim_{z \to a} \frac{(h \circ f)(z) - (h \circ f)(a)}{z - a} &= \lim_{z \to a} \frac{(h \circ f)(z) - (h \circ f)(a)}{f(z) - f(a)} \cdot \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \\ &= h'(f(a)) \cdot f'(a), \end{aligned}$$

where we use that limits interchange with products (Fact 0.5.1) for the first equality. For the second equality, we use that f is continuous in a and thus $f(z) \to f(a)$ when $z \to a$. $\square$

Example 2.1.4. (1) Let $n \geqslant 0$. Then the function

$$f : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z^n \end{array}$$

is entire with derivative $f'(z) = nz^{n-1}$. (We leave the verification of this claim as an exercise.)

(2) Let $n < 0$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Then the function

$$f : \begin{array}{ccc} \mathbb{C}^\times & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z^n \end{array}$$

is entire with derivative $f'(z) = nz^{n-1}$. (We leave the verification of this claim as an exercise.)

(3) A *polynomial function* is a function $f : \mathbb{C} \to \mathbb{C}$ of the form $f(z) = \sum_{n=0}^{d} c_n z^n$ for some $c_0, \ldots, c_d \in \mathbb{C}$. A polynomial function is entire with derivative $f'(z) = \sum_{n=0}^{d-1} (n+1) c_{n+1} z^n$.

(4) Let $f, g : U \to \mathbb{C}$ be holomorphic and assume that $g(z) \neq 0$ for $z \in U$. Then $\frac{f}{g} : U \to \mathbb{C}$ is holomorphic with derivative

$$\left(\frac{f}{g}\right)'(z) = \frac{f'g - fg'}{g^2}(z).$$

A *rational function* is a function of the form $\frac{f}{g} : U \to \mathbb{C}$ for two polynomial functions f and g.

2.2 The Cauchy-Riemann equations

In this section, we investigate how complex differentiability relates to real differentiability. For this, we recall the concept of differentiability in two real variables.

Recall that $x+\mathbf{i}y = \binom{x}{y}$ under the identification $\mathbb{C} = \mathbb{R}^2$. Let $U \subset \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ a function. Then there exist unique functions $u, v : U \to \mathbb{R}$ such that

$$f(x+iy) \;=\; u(x,y)+\mathbf{i}v(x,y)$$

for all $x, y \in \mathbb{R}$; namely

$$u(x,y) \;=\; \mathrm{Re}\,(f(x+\mathbf{i}y)) \qquad \text{and} \qquad v(x,y) \;=\; \mathrm{Im}\,(f(x+\mathbf{i}y)).$$

We write $f = u+\mathbf{i}v$ for short to express these relations.

Definition 2.2.1. Let $a = \binom{b}{c} \in U$. The function $f : U \to \mathbb{R}^2$ is *(real) differentiable in a* if there is a real 2×2-matrix $J_f(a)$ such that

$$\lim_{\binom{x}{y}\to\binom{b}{c}} \frac{\left(f(x,y)-f(b,c)-J_f(a)\cdot\left(\binom{x}{y}-\binom{b}{c}\right)\right)}{\left\|\binom{x}{y}-\binom{b}{c}\right\|} \;=\; \begin{pmatrix}0\\0\end{pmatrix}.$$

The function $f : U \to \mathbb{R}^2$ is *differentiable* if it is differentiable in all $a \in U$.

The matrix J_f is unique if it exists, and it is called the *Jacobian matrix of f*. Its coefficients are denoted as follows:

$$J_f(a) \;=\; \begin{pmatrix}\frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a)\\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a)\end{pmatrix} \;=\; \begin{pmatrix}u_x(a) & u_y(a)\\ v_x(a) & v_y(a)\end{pmatrix}.$$

The key lemma to relate complex and real differentiation is the following.

Lemma 2.2.2. *Let A be a real 2×2-matrix and $w = r+\mathbf{i}s \in \mathbb{C}$. Then $wz = A\cdot\binom{x}{y}$ for all $z = x+\mathbf{i}y \in \mathbb{C}$ if and only if $A = \left(\begin{smallmatrix} r & -s\\ s & r\end{smallmatrix}\right)$.*

Proof. The computation

$$wz \;=\; (r+\mathbf{i}s)\cdot(x+\mathbf{i}y) \;=\; (rx-sy)+\mathbf{i}(ry+sx) \;=\; \binom{rx-sy}{ry+sx} \;=\; \begin{pmatrix} r & -s\\ s & r\end{pmatrix}\binom{x}{y}$$

shows that $z \mapsto wz$ is an $\mathbb{R}$-linear map $\mathbb{R}^2 \to \mathbb{R}^2$ that corresponds to the (uniquely determined) matrix $A = \left(\begin{smallmatrix} r & -s\\ s & r\end{smallmatrix}\right)$. $\square$

Theorem 2.2.3. *Let $U \subset \mathbb{C}$ be an open subset and $f = u+\mathbf{i}v : U \to \mathbb{C}$ a function. Then f is holomorphic if and only if f is differentiable and if the* Cauchy-Riemann equations *hold:*

$$u_x \;=\; v_y \qquad \textit{and} \qquad u_y \;=\; -v_x.$$

If f is holomorphic, then $f'(a) = u_x(a)+\mathbf{i}v_x(a)$ for all $a \in U$.

Proof. Let $a = \binom{b}{c} \in U$. The function f is complex differentiable in a if and only if

$$\frac{f(z)-f(a)-f'(a)\cdot(z-a)}{|z-a|} \underset{z\to a}{\longrightarrow} 0$$

with $f'(a) = r + \mathbf{i}s$ for some $r,s \in \mathbb{R}$. By Lemma 1.2.2, this is equivalent to

$$\frac{f(x,y)-f(b,c)-J_f(a)\cdot(\binom{x}{y}-\binom{b}{c})}{\|\binom{x}{y}-\binom{b}{c}\|} \underset{\binom{x}{y}\to\binom{b}{c}}{\longrightarrow} 0$$

with $J_f(a) = \begin{pmatrix} r & -s \\ s & r \end{pmatrix}$ for some $r,s \in \mathbb{R}$.

By the definition of the partial differentials u_x, u_y, v_x and v_y, we have $r = u_x(a) = v_y(a)$ and $s = -u_x(a) = v_y(a)$. Thus f is holomorphic if and only if f is (real) differentiable and if the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold. □

Example 2.2.4. (1) The function

$$\begin{array}{rccc} f: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & z^2 \end{array}$$

satisfies

$$f(x+\mathbf{i}y) \;=\; (x+\mathbf{i}y)^2 \;=\; (x^2-y^2)+\mathbf{i}(2xy) \;=\; u(x,y)+\mathbf{i}v(x,y)$$

for $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$, which satisfy Cauchy-Riemann equations:

$$u_x \;=\; 2x \;=\; v_y \qquad \text{and} \qquad u_y \;=\; -2y \;=\; -v_x.$$

This re-establishes that $z \mapsto z^2$ is an entire function.

(2) The complex conjugation

$$\begin{array}{rccc} f: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \bar{z} \end{array}$$

satisfies

$$f(x+\mathbf{i}y) \;=\; x-\mathbf{i}y \;=\; u(x,y)+\mathbf{i}v(x,y)$$

for $u(x,y) = x$ and $v(x,y) = -y$. Since $u_x = 1 \neq -1 = v_y$, the complex conjugation is not holomorphic.

Theorem 2.2.5 (Inverse function theorem)**.** *Let U and V be open subsets of $\mathbb{C}$ and $f : U \to V$ a holomorphic bijection with inverse bijection $g : V \to U$. If g is continuous and if $f'(a) \neq 0$ for all $a \in U$, then g is holomorphic with derivative*

$$g'(b) \;=\; \frac{1}{f'(g(b))}$$

for all $b \in V$.

Proof. Consider $w, b \in V$ with images $z = g(w)$ and $a = g(b)$. Since both f and g are continuous, w converges to b if and only if z converges to a (where we consider w and z as variables). Since $f'(a) \neq 0$, we can exchange the limit with $(-)^{-1}$ (Fact 0.5.1), which yields

$$g'(b) = \lim_{w \to b} \frac{g(w) - g(b)}{w - b} = \lim_{z \to a} \frac{z - a}{f(z) - f(a)} = \left(\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \right)^{-1} = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

This shows, in particular, that g is complex differentiable in b. □

Remark 2.2.6. The assumption that g is continuous is not necessary for Theorem 1.2.5 to hold. This can be seen by applying the implicit function theorem of (real) multivariate analysis, which shows that g is (real) differentiable and therefore continuous. We will show this later on with an argument from complex analysis.

2.3 Power series

Definition 2.3.1. A *(complex) power series* is an expression of the form

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_0, a_1, \ldots \in \mathbb{C}$. When the context is clear, we write $\sum a_n z^n$ for short.

In the following we investigate the question for which $z \in \mathbb{C}$ such a power series converges and write $\sum a_n z^n = f(z)$ if $\sum a_n z^n$ converges to $f(z)$ (where $f(z)$ is an expression in z). Subsequently, we show that power series define holomorphic functions whenever they converge on an open disc of the form $D_r(a) = \{z \in \mathbb{C} \mid |z - a| < r\}$.

Lemma 2.3.2. *The* geometric series $\sum z^n$ *converges absolutely*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

for all $z \in D_1(0)$.

Proof. Consider the partial sums

$$S_n = \sum_{i=1}^{n} |z^i| = \frac{1 - |z|^{n+1}}{1 - |z|}$$

of absolute values where the identity on the left hand side results from multiplying both sides by $1 - |z|$. For $z \in D_1(0)$, we have $\lim_{n \to \infty} |z|^{n+1} = 0$. Thus

$$\sum_{n=0}^{\infty} |z|^n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - |z|^{n+1}}{1 - |z|} = \frac{1}{1 - |z|}$$

converges. In other words, $\sum z^n$ converges absolutely for $z \in D_1(0)$. The same computation with $|z|$ replaced by z shows that $\sum z^n$ converges to $\frac{1}{1-z}$ for $z \in D_1(0)$. □

As a consequence, we see that the power series $\sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function $f: D_1(0) \to \mathbb{C}$. In fact, $\frac{1}{1-z}$ extends to a holomorphic function $f: \mathbb{C} \setminus \{1\} \to \mathbb{C}$, which is a first example for analytic continuation, which we discuss in a later section, cf. Definition 4.1.1 and section 5.1.

Definition 2.3.3. The *radius of convergence* of a power series $\sum a_n z^n$ is

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

where we define $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Theorem 2.3.4 (Cauchy-Hadamard). *Let $\sum a_n z^n$ be a power series with radius of convergence R. Then $\sum a_n z^n$ converges absolutely if $|z| < R$ and it diverges if $|z| > R$.*

Proof. Let $L = \limsup |a_n|^{1/n} = \frac{1}{R}$. Assume that $|z| < R$ or, equivalently, that $L \cdot |z| < 1$. Then there is an $\epsilon > 0$ such that $q = (L+\epsilon) \cdot |z| < 1$. By the definition of $L = \limsup |a_n|^{1/n}$, there is an $N \geqslant 0$ such that $|a_n|^{1/n} \leqslant L + \epsilon$ for all $n \geqslant N$. Thus

$$\sum_{n=0}^{\infty} |a_n| \cdot |z|^n \leqslant C + \sum_{n=0}^{\infty} \big((L+\epsilon) \cdot |z|\big)^n = C + \sum_{n=0}^{\infty} q^n$$

for some sufficiently large $C \in \mathbb{R}$. Since $q < 1$, Lemma 1.3.2 implies that $\sum q^n$ converges, which shows that $\sum a_n z^n$ converges absolutely for $|z| < R$.

Assume that $|z| > R$ or, equivalently, that $L \cdot |z| > 1$. It suffices to show that the sequence of partial sums $S_n = \sum_{n=0}^{n} a_n z^n$ is not a Cauchy sequence, in order to show that $\sum a_n z^n$ converges. First note that there is an $\epsilon > 0$ such that $q = (L - \epsilon) \cdot |z| > 1$. By the definition of $L = \limsup |a_n|^{1/n}$, there are infinitely many $n \in \mathbb{N}$ such that $|a_n|^{1/n} \geqslant L - \epsilon$. For such n, we have

$$|S_n - S_{n-1}| = |a_n z^n| \geqslant \big((L-\epsilon) \cdot |z|\big)^n = q^n,$$

which does not tend to 0 for increasing n. This shows that $\{S_n\}$ is not a Cauchy sequence and that $\sum a_n z^n$ diverges. □

Remark 2.3.5. For $|z| = R$, it depends on the series and the specific z whether $\sum a_n z^n$ converges or diverges.

Example 2.3.6. (1) The *exponential series*

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

has radius of convergence

$$R = \frac{1}{\limsup |\frac{1}{n!}|^{1/n}} = \frac{1}{\frac{1}{\infty}} = \infty$$

since

$$|n!|^{1/n} \geqslant \big((\tfrac{n}{2})^{n/2}\big)^{1/n} = \sqrt{\tfrac{n}{2}} \xrightarrow[n\to\infty]{} \infty.$$

This defines the exponential function $\exp: \mathbb{C} \to \mathbb{C}$ with $\exp(z) = e^z$, where we use the notation $\exp(z)$ and e^z interchangeably.

(2) Both

$$\cos z = \sum_{n\in\mathbb{N}\text{ even}} \frac{\mathbf{i}^n}{n!} z^n \qquad \text{and} \qquad \sin z = \sum_{n\in\mathbb{N}\text{ odd}} \frac{\mathbf{i}^{n-1}}{n!} z^n$$

have radius of convergence $R=\infty$ and thus define functions $\cos : \mathbb{C}\to\mathbb{C}$ and $\sin : \mathbb{C}\to\mathbb{C}$.

Theorem 2.3.7. *Let $\sum a_n z^n$ be a power series with radius of convergence R. Then the function*

$$\begin{array}{rccc} f: & D_R(0) & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum a_n z^n \end{array}$$

is holomorphic with derivative

$$f'(z) = \sum_{n=0}^{\infty} (n+1)\cdot a_{n+1} z^n,$$

which is a power series with the same radius of convergence R.

Proof. For the sake of this proof, let $g(z) = \sum_{n=0}^{\infty}(n+1)\cdot a_{n+1}z^n$. Using $m=n+1$ and Exercise 0.8.2, we have

$$\begin{aligned} \limsup |m\cdot a_m|^{1/(m-1)} &= \underbrace{\limsup m^{1/(m-1)}}_{=1} \cdot \limsup \left(|a_m|^{\frac{1}{m}}\right)^{m/(m-1)} \\ &= \limsup |a_m|^{1/m} = \frac{1}{R}, \end{aligned}$$

which shows that the radius of convergence of the power series $\sum(n+1)\cdot a_{n+1}z^n$ is equal to that of $\sum a_n z^n$, which defines g as a function from $D_R(0)$ to $\mathbb{C}$.

In order to prove that $\sum a_n z^n$ is holomorphic on $D_R(0)$ with derivative $f'(z)=g(z)$, we divide the power series into its partial sum and its "tail"

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \qquad \text{and} \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n,$$

respectively. Consider a fixed element $z\in D_R(0)$ and $|z|<r<R$. For an element $h\in D_{r-|z|}(0)\setminus\{0\}$, we compute

$$\begin{aligned} &\frac{f(z+h)-f(z)}{h} - g(z) \\ &= \left(\frac{S_N(z+h)-S_N(z)}{h} - S_N'(z)\right) + \left(S_N'(z)-g(z)\right) + \left(\frac{E_N(z+h)-E_N(z)}{h}\right) \end{aligned}$$

We have

$$\frac{S_N(z+h)-S_N(z)}{h} - S_N'(z) \underset{h\to 0}{\longrightarrow} 0,$$

by definition of the derivative $S_N'(z)$, and

$$S_N'(z) - g(z) \; = \; -\sum_{n=N+1}^{\infty} (n+1)\cdot a_{n+1} z^n \underset{N\to\infty}{\longrightarrow} 0.$$

Since $a^n - b^n = (a-b)\cdot\sum_{i=0}^{n-1} a^i b^{n-1-i}$ for $a = z+h$ and $b = z$, we have for small h (for which $|z| < r$ and $|z+h| < r$):

$$\begin{aligned}
\left|\frac{E_N(z+h)-E_N(z)}{h}\right| &= \frac{1}{|h|}\cdot\left|\sum_{n=N+1}^{\infty} a_n\cdot\big((z+h)^n - z^n\big)\right| \\
&= \frac{1}{|h|}\cdot\left|\sum_{n=N+1}^{\infty} a_n\cdot(z+h-z)\cdot\sum_{i=0}^{n-1}(z+h)^i z^{n-1-i}\right| \\
&\leqslant \frac{1}{|h|}\cdot\sum_{n=N+1}^{\infty} |a_n|\cdot|z+h-z|\cdot\left|\sum_{i=0}^{n-1}(z+h)^i z^{n-1-i}\right| \\
&\leqslant \left|\frac{h}{h}\right|\cdot\sum_{n=N+1}^{\infty}|a_n|\cdot\sum_{i=0}^{n-1}\left|(z+h)^i z^{n-1-i}\right| \\
&\leqslant \sum_{n=N+1}^{\infty}|a_n|\cdot n\cdot r^{n-1},
\end{aligned}$$

which tends to 0 as $N\to\infty$. Thus

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \;\leqslant\; \left|S_N'(z)-g(z)\right| + \sum_{n=N+1}^{\infty}|a_n|\cdot n\cdot r^{n-1} \underset{N\to\infty}{\longrightarrow} 0,$$

which shows that

$$g(z) \; = \; \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

is the differential $f'(z)$ of f in z. This shows, in particular, that f is holomorphic as a function in $D_R(z)$. □

As an immediate consequence, we have:

Corollary 2.3.8. *The functions* exp, cos *and* sin *are entire.*

Proposition 2.3.9. *Consider two power series $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ that are absolutely convergent on $D_r(0)$ with $r > 0$ (possibly $r = \infty$) and let $c \in \mathbb{C}$. Then*

$$\begin{aligned}
(f\pm g)(z) &= \sum_{n=0}^{\infty}(a_n \pm b_n) z^n; \\
(c\cdot f)(z) &= \sum_{n=0}^{\infty}(c\cdot a_n) z^n; \\
(f\cdot g)(z) &= \sum_{n=0}^{\infty}\Big(\sum_{i=0}^{n} a_i b_{n-i}\Big) z^n \qquad \textit{(Cauchy product)}
\end{aligned}$$

for all $z \in D_r(0)$. In particular, all power series converge absolutely for $z \in D_r(0)$.

Proof. All three formulas follow by proving them for the partial sums of the corresponding power series and then letting the number of terms in the partial sums go to infinity. Since the limit of the partial sums converges, the power series in these equations have radius of convergence $R > r$ and thus converge absolutely for $z \in D_r(0)$. We omit the details. $\square$

Corollary 2.3.10. *For all complex numbers $z = x + \mathbf{i}y$ and w, we have*

(1) $\exp'(z) = \exp(z)$	(4) $e^{iz} = \cos z + i \sin z$	(7) $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$
(2) $\sin'(z) = \cos(z)$	(5) $e^{z} = e^{x}(\cos y + i \sin y)$	(8) $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
(3) $\cos'(z) = -\sin(z)$	(6) $e^{z+w} = e^{z} \cdot e^{w}$	(9) $z = \lvert z\rvert \cdot e^{i \arg z}$

The proof of these identifies follow from a direct manipulation of the corresponding power series, which is left as an exercise. The identity $e^{iz} = \cos z + i \sin z$ is known as *Euler's formula*.

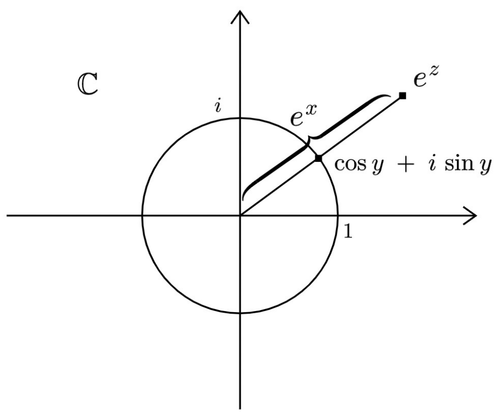
2.4 The logarithm

The real exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is a bijection with inverse $\log : \mathbb{R}_{>0} \to \mathbb{R}$. In this section, we investigate the corresponding behaviour of the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$.

As a first observation, we compute for $z = x + \mathbf{i}y \in \mathbb{C}$ that

$$e^{z} \;=\; \underbrace{e^{x}}_{\in \mathbb{R}_{>0}} \cdot \underbrace{\big(\cos y + \mathbf{i} \sin y\big)}_{\in \mathbb{S}^1} \;\in \mathbb{C}^{\times}$$

where $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit sphere. Therefore the image of $\exp : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}^{\times} = \mathbb{R}_{>0} \times \mathbb{S}^1$.



If $w \in \mathbb{C}^{\times}$ has polar coordinates $(|w|, \arg w)$ with $|w| \in \mathbb{R}_{>0}$ and $\arg w \in \mathbb{R}/2\pi\mathbb{Z}$, then

$$\exp^{-1}(w) \;=\; \big\{x + \mathbf{i}y \in \mathbb{C} \,\big|\, x = \log|w|,\ y \in \arg w + 2\pi\mathbb{Z}\big\}.$$

In particular, $\exp^{-1}(1) = 2\pi\mathbf{i}\mathbb{Z}$. This shows that $\exp : \mathbb{C} \to \mathbb{C}^\times$ is *not* injective, in contrast to $\exp : \mathbb{R} \to \mathbb{R}_{>0}$.

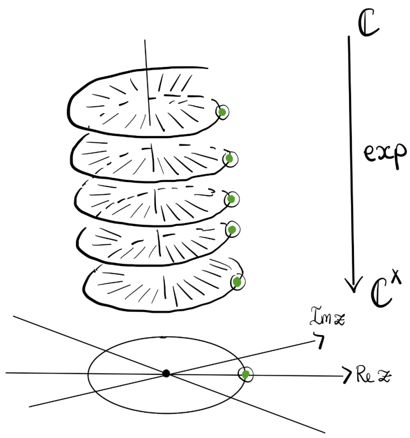
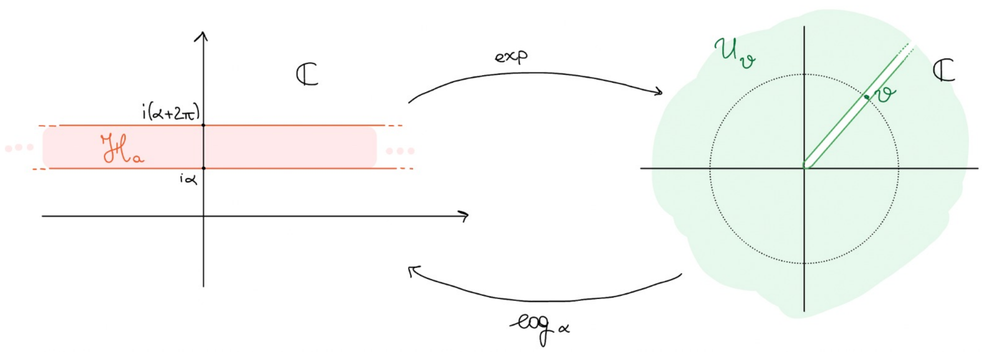


Figure 2.1: Enter Caption

Idea: restrict exp to a bijection between subsets of the forms

$$\mathcal{H}_\alpha \;=\; \{z \in \mathbb{C} \mid \operatorname{Im}(z) \in (\alpha, \alpha + 2\pi)\} \qquad \text{and} \qquad \mathcal{U}_\vartheta \;=\; \mathbb{C}^\times \setminus \{\lambda \cdot \vartheta \mid \lambda > 0\}$$

for $\alpha \in \mathbb{R}$ and $\vartheta = e^{\mathbf{i}\alpha} \in \mathbb{S}^1$.



Definition 2.4.1. The *α-branch of the logarithm* is the inverse $\log_\alpha : \mathcal{U}_\vartheta \to \mathcal{H}_\alpha$ to the bijection $\exp : \mathcal{H}_\alpha \to \mathcal{U}_\vartheta$ for $\alpha \in \mathbb{R}$ and $\vartheta = e^{\mathbf{i}\alpha}$. The *principal branch of* log is $\log = \log_{-\pi} : \mathcal{U}_{-1} \to \mathcal{H}_{-\pi}$.

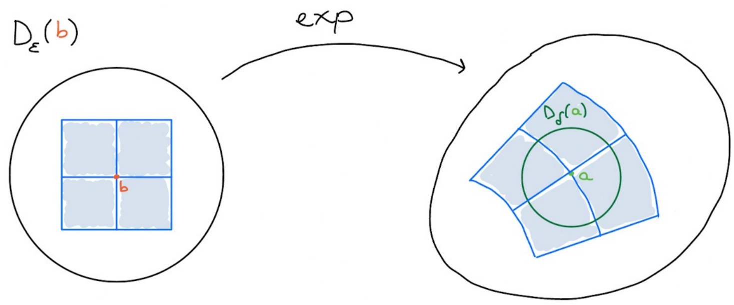
Proposition 2.4.2. *Let $\alpha \in \mathbb{R}$ and $\vartheta = e^{\mathbf{i}\alpha}$. The function $\log_\alpha : \mathcal{U}_\vartheta \to \mathcal{H}_\alpha$ is holomorphic with derivative*

$$\log_\alpha'(z) \;=\; \frac{1}{z}.$$

Proof. As a first step, we show that $\log_\alpha$ is continuous, i.e., for every $a \in \mathcal{U}_\vartheta$ with image $b = \log_\alpha(a)$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\log_\alpha\big(D_\delta(a) \cap \mathcal{U}_\vartheta\big) \;\subset\; D_\epsilon(b).$$

After making ϵ and δ suitably small, we can assume that $D_\delta(a) \subset \mathcal{U}_\vartheta$ and that $D_\epsilon(b) \subset \mathcal{H}_\alpha$. Since $\exp \circ \log_\alpha$ is the identity on $D_\delta(a)$, the above inclusion follows from the inclusion $D_\delta(a) \subset \exp(D_\epsilon(b))$ of the respective images under exp. Let Q be a small square around b contained in $D_\epsilon(b)$. Then the horizontal boundary lines of the square are mapped to straight lines that lie on rays originating from the origin of $\mathbb{C}$, and the vertical boundary lines of the square lie on circles around the origin. This makes clear that $\exp(Q)$ is a deformed square that contains a in its interior (as illustrated below) and thus also an open disc $D_\delta(a)$ for suitably small δ. This shows that $\log_\alpha$ is continuous.



Therefore we can apply the inverse function theorem (Theorem 1.2.5) to $\exp : \mathcal{H}_\alpha \to \mathcal{U}_\vartheta$, which shows that its inverse $\log_\alpha$ is holomorphic with derivative

$$\log_\alpha'(z) \;=\; \frac{1}{\exp'\big(\log_\alpha(z)\big)} \;=\; \frac{1}{\exp\big(\log_\alpha(z)\big)} \;=\; \frac{1}{z},$$

where we use Exercise 3 from List 2 for $\exp'(w) = \exp(w)$. □

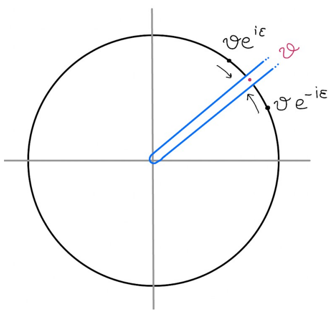
Remark 2.4.3. (1) We have

$$\log_\alpha(z) - \log_\beta(z) \;\in\; 2\pi\mathbf{i}\mathbb{Z}$$

for all $\alpha, \beta \in \mathbb{R}$ and $z \in \mathcal{U}_{e^{\mathbf{i}\alpha}} \cap \mathcal{U}_{e^{\mathbf{i}\beta}}$.

(2) For $\alpha \in \mathbb{R}$ and $\vartheta = e^{\mathbf{i}\alpha}$, we have

$$\lim_{\substack{\epsilon\to 0\\ \text{with } \epsilon>0}} \Big(\log_\alpha(\vartheta\cdot e^{-\mathbf{i}\epsilon}) - \log_\alpha(\vartheta\cdot e^{\mathbf{i}\epsilon})\Big) \;=\; 2\pi\mathbf{i}.$$



(3) We have $\log_\alpha(z\cdot w) \equiv \log_\alpha(z)+\log_\alpha(w) \pmod{2\pi\mathbf{i}\mathbb{Z}}$, but in general these terms are not equal. For example:

$$\begin{aligned}\log\left(e^{\frac{3}{4}\pi\mathbf{i}}\cdot e^{\frac{3}{4}\pi\mathbf{i}}\right) \;&=\; \log\left(e^{-\frac{1}{2}\pi\mathbf{i}}\right) \;=\; -\tfrac{1}{2}\pi\mathbf{i}\\ &\neq\; \tfrac{3}{2}\pi\mathbf{i} \;=\; \left(\tfrac{3}{4}\pi\mathbf{i}\right)+\left(\tfrac{3}{4}\pi\mathbf{i}\right) \;=\; \log\left(e^{\frac{3}{4}\pi\mathbf{i}}\right)+\log\left(e^{\frac{3}{4}\pi\mathbf{i}}\right).\end{aligned}$$

Insert an illustration

Application 1: Complex powers

Heuristically, we could attempt to define the power z^w for $z,w\in\mathbb{C}$ with $z,w\neq 0$ as

$$z^w \;\overset{?}{=}\; (e^{\log_\alpha(z)})^w \;=\; e^{w\log_\alpha(z)}$$

for suitable $\alpha\in\mathbb{R}$. However, this expression depends on the choice of α.

A better definition of z^w is as the set

$$z^w \;=\; \left\{e^{w\cdot\log_\beta(z)}\,\middle|\,\beta\in\mathbb{R},\ e^{\mathbf{i}\beta}\neq z/|z|\right\} \;=\; \left\{e^{w\cdot(\log_\alpha(z)+k\cdot 2\pi\mathbf{i})}\,\middle|\,k\in\mathbb{Z}\right\},$$

which does not depend on the choice of α, but rather combines all possible such choices.

Lemma 2.4.4. *Let $z,w\in\mathbb{C}$ with $z,w\neq 0$. Then the set z^w is a singleton if and only if $w\in\mathbb{Z}$ and it is finite if and only if $w\in\mathbb{Q}$.*

Proof. The set

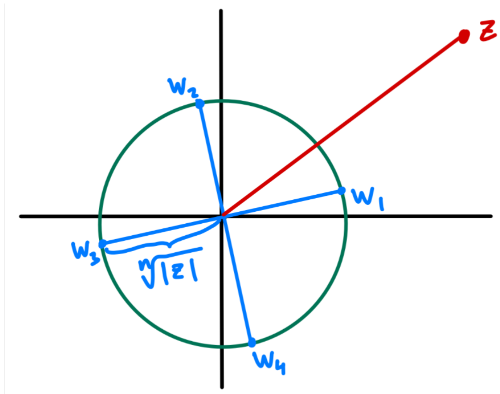
$$z^w \;=\; \left\{e^{w\cdot(\log_\alpha(z)+k\cdot 2\pi\mathbf{i})}\,\middle|\,k\in\mathbb{Z}\right\},$$

is finite if and only if $kw\in\mathbb{Z}$ for some $k>0$, which is the case if and only if $w\in\mathbb{Q}$. This proves the second claim. The first claim follows since z^w is a singleton if and only if $kw\in\mathbb{Z}$ for all $k>0$, which means that $w\in\mathbb{Z}$. □

Example 2.4.5. For positive $n\in\mathbb{N}$, we have

$$\sqrt[n]{z} \;=\; z^{1/n} \;=\; \Big\{\underbrace{e^{\frac{1}{n}\log_\alpha(z)}}_{=\rho_0}\cdot\underbrace{e^{\frac{k}{n}\cdot 2\pi\mathbf{i}}}_{=\zeta_n^k}\,\Big|\,k=1,\dots,n\Big\}.$$

where ρ_0 is an n-th root of z (which depends on the choice of α) and $\zeta_n = e^{\frac{2\pi\mathbf{i}}{n}}$ is a *primitive n-th root of unity*.



Proposition 2.4.6. *Let $w \in \mathbb{C}^\times$, $\alpha \in \mathbb{R}$ and $\vartheta = e^{\mathbf{i}\alpha}$. Then the map*

$$\begin{array}{rccc} f: & \mathcal{U}_\vartheta & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & e^{w\cdot\log_\alpha(z)} \end{array}$$

is holomorphic with derivative

$$f'(z) \;=\; w\cdot e^{(w-1)\cdot\log_\alpha(z)}.$$

If we interpret $f(z)$ as (a branch of) z^w, then Proposition 1.4.6 shows that $f'(z)$ is (a branch of) $w\cdot z^{w-1}$.

Proof. As a composition of holomorphic functions, f is holomorphic. We use the rules from Proposition 1.1.3 to compute:

$$\begin{aligned} f'(z) \;&=\; \exp'(w\cdot\log_\alpha(z))\cdot w\cdot\log_\alpha'(z) \;=\; \exp(w\cdot\log_\alpha(z))w\cdot\frac{1}{z} \\ &=\; w\cdot e^{-1\cdot\log_\alpha(z)}\cdot e^{w\cdot\log_\alpha(z)} \;=\; w\cdot e^{(w-1)\cdot\log_\alpha(z)}. \end{aligned}$$

□

Application 2: Inverse trigonometric functions

Let $z=\cos(w)$. If we try to express w in dependency of z, we find that

$$\begin{aligned} & z \;=\; \cos(w) \;=\; \frac{1}{2}(e^{\mathbf{i}w}+e^{-\mathbf{i}w}) \\ \Leftrightarrow\; & 2z \;=\; e^{\mathbf{i}w}+e^{-\mathbf{i}w} \\ \Leftrightarrow\; & (e^{\mathbf{i}w})^2-2ze^{\mathbf{i}w}+1 \;=\; 0 \\ \Leftrightarrow\; & w \;=\; -\mathbf{i}\cdot\log_\alpha\left(z\pm\mathbf{i}\sqrt{1-z^2}\right) \end{aligned}$$

for a suitable $\alpha\in\mathbb{R}$. The *principal branch of* arccos is

$$\begin{array}{rccc} \arccos: & \mathbb{C}\setminus\{x\in\mathbb{R}\mid |x|\geqslant 1\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & -\mathbf{i}\cdot\log\left(z+\mathbf{i}\sqrt{1-z^2}\right) \end{array}$$

where $\alpha = -\pi$ and $1 - z^2 \in \mathcal{U}_{-1}$.

Insert an illustration

Similarly, we define the *principal branch of* arcsin as

$$\begin{array}{rccc} \arcsin: & \mathbb{C} \setminus \{x \in \mathbb{R} \mid |x| \geqslant 1\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & -\mathbf{i} \cdot \log\left(\mathbf{i}z + \sqrt{1-z^2}\right). \end{array}$$

Chapter 3

Path integrals

Motivation

Question. *Does the expression* $\int_a^b f(z)\,dz$ *make sense for* $f : U \to \mathbb{C}$ *and* $a, b \in U \subset \mathbb{C}$*?*

Thought 1. If $f = F'$ for a holomorphic function $F : U \to \mathbb{C}$, then we expect that

$$\int_a^b f(z)\,dz \;=\; F(b) - F(a).$$

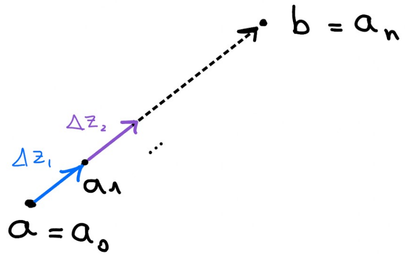
Thought 2. Attempt to construct F:

- Choose $F(a)$ arbitrarily.
- For small $\Delta z = b - a$, we expect that

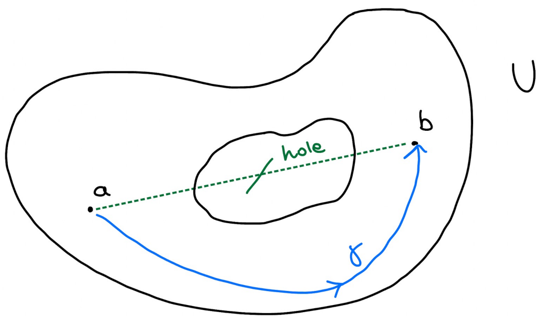
$$F(b) \;\approx\; F(a) + \Delta z \cdot f(a).$$

- For general b, we would like to define

$$F(b) \;=\; F(a) \;+\; \lim_{\substack{n\to\infty \\ \Delta z_i \to 0}} \;\sum_{i=1}^{n} \Delta z_i \cdot f(a_{i-1}).$$



Problem.



Thought 3. Integrate along path γ from a to b.

Question. *Is $\int_\gamma f(z)\,dz$ independent of γ?*

3.1 Preliminaries

Definition 3.1.1. Let $a \leqslant b$ be real numbers and $f : [a,b] \to \mathbb{C}$ continuous. We define

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re}\,(f(t))\,dt + \mathbf{i}\cdot\int_a^b \mathrm{Im}\,(f(t))\,dt$$

and

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.$$

Proposition 3.1.2. *Let $a \leqslant b$ be real numbers and $f : [a,b] \to \mathbb{C}$ continuous. Then the following hold.*

(1) $\int_a^b f(t)\,dt$ is $\mathbb{C}$-linear in f, i.e.,

$$\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$$

and

$$\int_a^b (c\cdot f)(t)\,dt = c\cdot\int_a^b f(t)\,dt$$

for all continuous $g : [a,b] \to \mathbb{C}$ and $c \in \mathbb{C}$.

(2) For $a = a_0 \leqslant \ldots \leqslant a_n = b$,

$$\int_a^b f(t)\,dt = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(t)\,dt$$

(3) If $f = F' = \binom{\partial F/\partial x}{\partial F/\partial y}$ for (real) differentiable $F : [a,b] \to \mathbb{C}$, then

$$\int_a^b f(t)\, dt \;=\; F(b) - F(a).$$

(4) Let $\alpha : [c,d] \to [a,b]$ be a continuously differentiable map where $c \leqslant d$. Then

$$\int_{\alpha(c)}^{\alpha(d)} f(t)\, dt \;=\; \int_c^d f(\alpha(s)) \cdot \alpha'(s)\, ds.$$

(5)

$$\left| \int_a^b f(t)\, dt \right| \quad \leqslant \quad \int_a^b |f(t)|\, dt \quad \leqslant \quad (b-a) \cdot \max\{|f(t)| \mid a \leqslant t \leqslant b\}.$$

Note that $\{|f(t)| \mid a \leqslant t \leqslant b\}$ is compact as the image of the compact interval $[a,b]$ under the continuous map $|f| : [a,b] \to \mathbb{R}$, and therefore its maximum exists.

3.2 Path integrals

Definition 3.2.1. Let U be an open subset of $\mathbb{C}$. A *path in U* is a continuous map $\gamma : [a,b] \to U$ where $a \leqslant b$. A path $\gamma : [a,b] \to U$ is *closed* if $\gamma(a) = \gamma(b)$. It is *smooth* if it is continuously differentiable. We denote its derivative by

$$\gamma'(t) \;=\; \frac{d}{dt}\,\gamma(t) \;=\; \begin{pmatrix} \frac{d}{dt}\,\mathrm{Re}\,(\gamma(t)) \\ \frac{d}{dt}\,\mathrm{Im}\,(\gamma(t)) \end{pmatrix}.$$

In the literature, paths are also called curves or arcs. Typically we parameterize paths $\gamma : I \to U$ by the *unit interval* $I = [0,1]$.

Definition 3.2.2. Let $U \subset \mathbb{C}$ be open, $\gamma : [a,b] \to U$ smooth and $f : U \to \mathbb{C}$ continuous. The *path integral of f along γ* is

$$\int_\gamma f \;=\; \int_\gamma f(z)\, dz \;:=\; \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

The *arc length of γ* is

$$\ell(\gamma) \;:=\; \int_a^b |\gamma'(t)|\, dt.$$

Example 3.2.3.

(1) Let $p \in U$. The *constant path based at* p is the path $\gamma : I \to U$ with $\gamma(t) = p$ for all $t \in I$. Its derivative is $\gamma'(t) = \binom{0}{0}$ for all $t \in I$. Therefore

$$\ell(\gamma) \;=\; \int_0^1 |\gamma'(t)|\, dt \;=\; 0 \qquad \text{and} \qquad \int_\gamma f \;=\; \int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt \;=\; 0$$

for any continuous function $f : U \to \mathbb{C}$.

(2) Let $p, q \in U$. The *linear path from* p *to* q is the path $\gamma : I \to U$ given by $\gamma(t) = p + t \cdot (q - p)$ for $t \in I$. Its derivative is $\gamma'(t) = q - p$ for all $t \in I$. Therefore

$$\ell(\gamma) \;=\; \int_0^1 |q-p|\, dt \;=\; |q-p| \qquad \text{and} \qquad \int_\gamma f \;=\; \int_0^1 f(\gamma(t)) \cdot (q-p)\, dt$$

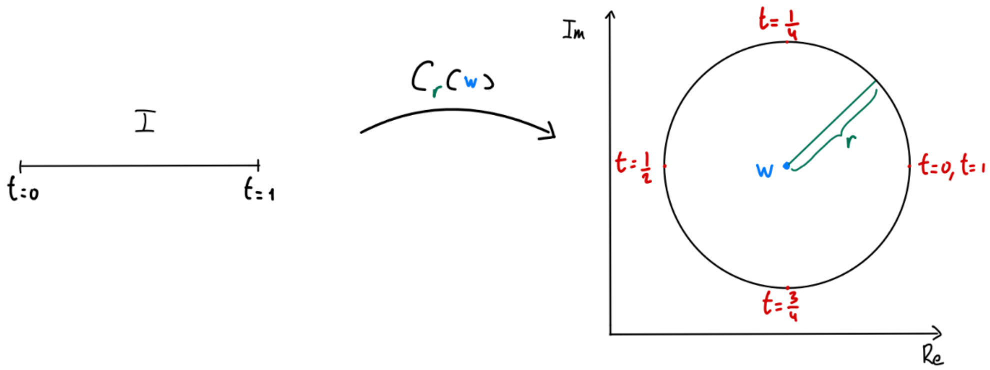
for any continuous function $f : U \to \mathbb{C}$. For example, if $\gamma(t) = t$ is the linear path from $0 = \gamma(0)$ to $1 = \gamma(1)$, then

$$\int_\gamma f \;=\; \int_0^1 \mathrm{Re}\,(f(t))dt$$

is the same as the usual integral of the real valued function $\mathrm{Re} \circ f : I \to \mathbb{R}$.

(3) Let $w \in \mathbb{C}$ and $r \geqslant 0$. The *(parameterized) circle of radius* r *around* w is the path

$$\begin{aligned} C_r(w) : \; I &\longrightarrow \mathbb{C} \\ t &\longmapsto w + re^{2\pi\mathbf{i}\cdot t} \end{aligned}$$



Its derivative is $C_r(w)'(t) = 2\pi\mathbf{i} \cdot re^{2\pi\mathbf{i}\cdot t}$ and its length is

$$\ell(C_r(w)) \;=\; \int_0^1 |2\pi\mathbf{i} \cdot re^{2\pi\mathbf{i}\cdot t}|\, dt \;=\; \int_0^1 2\pi \cdot r \cdot \underbrace{|e^{2\pi\mathbf{i}\cdot t}|}_{=1}\, dt \;=\; 2\pi \cdot r.$$

Let $w = 0$ and $r = 1$. Consider $f : \mathbb{C}^\times \to \mathbb{C}$ with $f(z) = z^n$ for $n \in \mathbb{Z}$. If $n \neq -1$, then $f = F'$ for $F(z) = \frac{1}{n+1} z^{n+1}$ and thus

$$\int_{C_1(0)} f \;=\; \int_{C_1(0)} z^n\,dz \overset{(!)}{=} F(1) - F(1) \;=\; 0,$$

as we show in Proposition 2.2.4. For $n = -1$, this integral is more interesting:

$$\int_{C_1(0)} \frac{1}{z}\,dz \;=\; \int_0^{2\pi} \frac{1}{e^{\mathbf{i}t}} \cdot \left(\frac{d}{dt}\,e^{\mathbf{i}t}\right) dt \;=\; \int_0^{2\pi} \frac{1}{e^{\mathbf{i}t}} \cdot \mathbf{i} \cdot e^{\mathbf{i}t}\,dt \;=\; \int_0^{2\pi} \mathbf{i}\,dt \;=\; 2\pi\mathbf{i}.$$

Proposition 3.2.4. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous and $\gamma : [a,b] \to U$ a smooth path. Then*

(1) $\int_\gamma f$ is $\mathbb{C}$-linear in f.

(2) For $a = a_0 \leqslant \ldots \leqslant a_n = b$ and $\gamma_i = \gamma|_{[a_{i-1},a_i]}$,

$$\int_\gamma f \;=\; \sum_{i=1}^{n} \int_{\gamma_i} f.$$

(3) If $f = F'$ for a holomorphic function $F : U \to \mathbb{C}$, then

$$\int_\gamma f \;=\; F(\gamma(b)) - F(\gamma(a)).$$

(4) Let $\alpha : [c,d] \to [a,b]$ be continuously differentiable with $\alpha(c) = a$ and $\alpha(d) = b$. Then

$$\int_{\gamma\circ\alpha} f \;=\; \int_\gamma f.$$

(5)

$$\left|\int_\gamma f\right| \;\leqslant\; \ell(\gamma) \cdot \max\{|f(z)| \mid z \in \operatorname{im}(\gamma)\}.$$

(6) Let $\gamma^- : [a,b] \to U$ be defined by $\gamma^-(t) = \gamma(a+b-t)$. Then

$$\int_{\gamma^-} f \;=\; -\int_\gamma f.$$

Proof. Properties (1) and (2) follow directly from the corresponding properties of Proposition 2.1.2.

Property (3) follows from the direct computation

$$\int_\gamma f \;=\; \int_a^b F'(\gamma(t)) \cdot \gamma'(t)\,dt \;=\; \int_a^b (F \circ \gamma)'(t)\,dt \;=\; F(\gamma(b)) - F(\gamma(a))$$

where we use the chain rule for multivariate real-valued functions for the second equality and Proposition 2.1.2.(3) for the third equality.

Property (4) follows from the direct computation

$$\begin{aligned}\int_{\gamma\circ\alpha} f &= \int_c^d f(\gamma\circ\alpha(s))\cdot(\gamma\circ\alpha)'(s)\,ds\\ &= \int_c^d f(\gamma(\alpha(s)))\cdot\gamma'(\alpha(s))\cdot\alpha'(s)\,ds\\ &= \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt = \int_\gamma f\end{aligned}$$

where we use Proposition 2.1.2.(4) for the third equality.

Property (5) follows from the direct computation

$$\begin{aligned}\left|\int_\gamma f\right| &= \left|\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt\right|\\ &\leqslant \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,dt\\ &\leqslant \ell(\gamma)\cdot\max\{|f(z)| \mid z\in\operatorname{im}(\gamma)\}\end{aligned}$$

where we use Proposition 2.1.2.(5) for the first inequality. The second inequality follows from $|f(\gamma(t))| \leqslant \max\{|f(z)| \mid z\in\operatorname{im}(\gamma)\}$.

Property (6) follows analogously to (4), using that $\gamma^- = \gamma\circ\alpha$ for the map $\alpha : [a,b]\to[a,b]$ with $\alpha(t) = a+b-t$. The difference is that the end points are reversed, i.e., $\alpha(a) = b$ and $\alpha(b) = a$, which leads to

$$\int_{\gamma^-} f = \int_{\alpha(a)}^{\alpha(b)} f(\gamma(t))\cdot\gamma'(f)\,dt = -\int_a^b f. \qquad \square$$

Definition 3.2.5. Let $U\subset\mathbb{C}$ be open. A *contour in U* is a path $\gamma : [a,b]\to U$ that is *piecewise smooth*, i.e. there are $a = a_0 < \cdots < a_n = b$ such that the restrictions $\gamma_i : [a_{i-1},a_i]\to U$ of γ to $[a_{i-1},a_i]$ are smooth for all $i = 1,\ldots,n$.

Given a contour $\gamma : [a,b]\to U$ and $a_0 < \cdots < a_n$ as above and a holomorphic function $f : U\to\mathbb{C}$, we define the *path integral of f along γ* as

$$\int_\gamma f = \sum_{i=1}^n \int_{\gamma_i} f$$

Remark 3.2.6. There is a unique minimal choice of $a_0 < \cdots < a_n$ for a contour γ, which are the points where γ fails to be continuously differentiable. By property (2) of

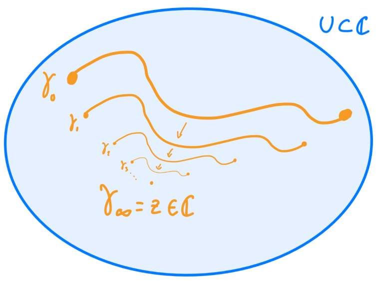
Proposition 2.2.4, the value of the path integral $\int_\gamma f$ is, however, independent of adding points to the sequence $a_0 < \cdots < a_n$, which shows that $\int_\gamma f$ is independent of the choice of $a_0 < \cdots < a_n$.

All properties of Proposition 2.2.4 generalize literally to contours; cf. Exercise 2.8.8.

Outlook: Cauchy's integral theorem

Our goal in this chapter is to prove the following.

Theorem (Cauchy's integral theorem)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\gamma : I \to \mathbb{C}$ a path. If γ is contractible, then $\int\limits_\gamma f = 0$.*



Question.

- *What does it mean that a path is contractible?*
- *What is $\int\limits_\gamma f$ for a path γ that is not smooth?*

Strategy. The proof of Cauchy's integral theorem is intricate and progresses along the following steps:

(1) Goursat: $\int_\gamma f = 0$ for triangular paths.

(2) Primitives for f on discs.

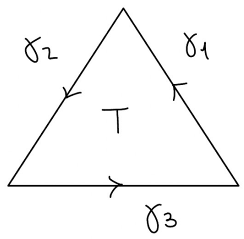
(3) Define $\int_\gamma f$ for arbitrary paths.

(4) Homotopy invariance of $\int_\gamma f$.

(5) Cauchy's integral theorem and more.

3.3 Goursat's theorem

A triangle in $\mathbb{C}$ is a subset of the following form:

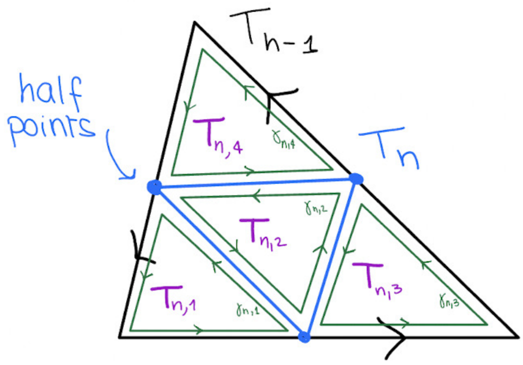


A boundary curve for T is a closed path $\gamma : I \to T$ that transverses once counter-clockwise around T (i.e., it is injective on $(0,1)$ and linear on the segments between the vertices of T). Note that γ is a contour and thus

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f.$$

Theorem 3.3.1 (Goursat)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $T \subset U$ a triangle with boundary curve $\gamma : I \to T$. Then $\int_\gamma f = 0$.*

Proof. Define $T_0 = T$ and $\gamma_0 = \gamma$. For $n \geqslant 1$, we define triangles T_n by recursion:



We subdivide a given triangle T_{n-1} into four congruent triangles $T_{n,1}, \ldots, T_{n,4}$ of equal size with respective boundary curves $\gamma_{n,1}, \ldots, \gamma_{n,4}$. Then

$$\int_{\gamma_{n-1}} f = \sum_{i=1}^{4} \int_{\gamma_{n,i}} f$$

We define $T_n = T_{n,k}$ and $\gamma_n = \gamma_{n,k}$ for a $k \in \{1, \ldots, 4\}$ such that

$$\left| \int_{\gamma_{n,k}} f \right| \geqslant \left| \int_{\gamma_{n,i}} f \right|$$

for all $i = 1, \ldots, 4$. Then there is a unique $z_0 \in T$ that is contained in T_n for all n, and

$$\ell(\gamma_n) = (\tfrac{1}{2})^n \cdot \ell(\gamma_0) \qquad \text{and} \qquad d(T_n) = (\tfrac{1}{2})^n \cdot d(T_0)$$

where $d(T_n) := \max\{|z-w| \mid z,w \in T_n\}$ is the diameter of T_n. We have

$$h(z) \;:=\; \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \quad \underset{z\to z_0}{\longrightarrow} \quad 0.$$

Then

$$\int_{\gamma_n} f \;=\; \int_{\gamma_n} \Big(f(z_0) + f'(z_0)(z-z_0) + h(z)(z-z_0) \Big)\, dz.$$

Since $f(z_0)+f'(z_0)(z-z_0)$ is linear in z, it has a primitive, and thus its path integral along γ_n is 0. Therefore

$$\int_{\gamma_n} f \;=\; \int_{\gamma_n} h(z)(z-z_0)\, dz.$$

We conclude that

$$\begin{aligned} \Big| \int_{\gamma} f \Big| \;\leqslant\; 4^n \cdot \Big| \int_{\gamma_n} f \Big| \;&\leqslant\; 4^n \cdot \int_{\gamma_n} |h(z)| \cdot |z-z_0|\, dz \\ &\leqslant\; 2^n \cdot d(T_n) \cdot 2^n \cdot \ell(\gamma_n) \cdot \max\{|h(z)| \mid z \in \mathrm{im}(\gamma_n)\} \\ &\leqslant\; d(T) \cdot \ell(\gamma) \cdot \max\{|h(z)| \mid z \in \mathrm{im}(\gamma_n)\} \quad \underset{n\to\infty}{\longrightarrow} \quad 0 \end{aligned}$$

where we use that $|z-z_0| \leqslant d(T_n)$ and Proposition 2.2.4.(5) in the third inequality. We conclude that $\left|\int_\gamma f\right| = 0$ and thus $\int_\gamma f = 0$, as claimed. □

3.4 Primitives on discs

Definition 3.4.1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ a function. A *primitive of f* is a holomorphic function $F : U \to \mathbb{C}$ with $F'(z) = f(z)$ for all $z \in U$.

Theorem 3.4.2. *Let $D_r(z_0)$ be an open disc of radius $r > 0$ with center z_0 and $f : D_r(z_0) \to \mathbb{C}$ a holomorphic function. For every $z \in D_r(z_0)$ define the linear path*

$$\gamma_z : \begin{array}{ccc} I & \longrightarrow & D_r(z_0) \\ t & \longmapsto & z_0 + t(z-z_0). \end{array}$$

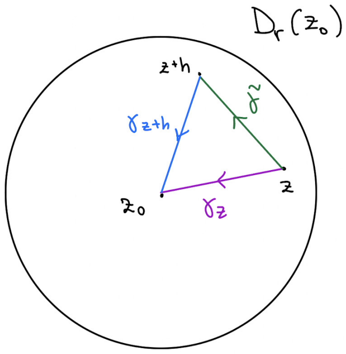
Then the function

$$F : \begin{array}{ccc} D_r(z_0) & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \int_{\gamma_z} f \end{array}$$

is a primitive of f.

Proof. We want to show that

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \;=\; f(z).$$



By Theorem 2.3.1, we have

$$F(z+h)-F(z) \;=\; \int_{\gamma_{z+h}} f - \int_{\gamma_z} f \;=\; \int_{\tilde{\gamma}} f$$

where $\tilde{\gamma}$ is the linear path from z to $z+h$:

Since f is continuous,

$$g(w) \;:=\; f(w)-f(z) \;\underset{w\to z}{\longrightarrow}\; 0$$

for fixed z. Since

$$\left| \int_{\tilde{\gamma}} g(w)\,dw \right| \;\leqslant\; \ell(\tilde{\gamma})\cdot \max\{|g(w)| \mid w\in \operatorname{im}(\tilde{\gamma})\},$$

and $\ell(\tilde{\gamma}) = |h|$, we have

$$\lim_{h\to 0} \frac{1}{h}\cdot \int_{\tilde{\gamma}} g(w)\,dw \;=\; 0$$

and

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} \;=\; \lim_{h\to 0} \frac{1}{h}\cdot \int_{\tilde{\gamma}} (f(z)+g(w))\,dw \;=\; \lim_{h\to 0} \frac{1}{h}\cdot h\cdot f(z) \;=\; f(z),$$

as desired. □

As a consequence, we can prove Cauchy's integral theorem for discs at once.

Corollary 3.4.3. *Let $f : D_r(z_0) \to \mathbb{C}$ be holomorphic and $\gamma : I \to D_r(z_0)$ a closed and smooth path. Then $\int_\gamma f = 0$.*

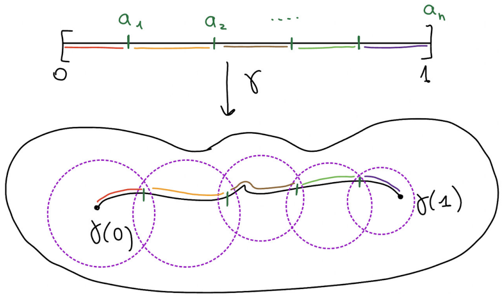
Proof. By Theorem 2.4.2, f has a primitive F. By Proposition 2.2.4.(3),

$$\int_\gamma f \;=\; F(\gamma(1)) - F(\gamma(0)) \;=\; 0.$$ □

3.5 Integrals along continuous paths

Idea. If a holomorphic function $f : U \to \mathbb{C}$ has a primitive $F : U \to \mathbb{C}$, then $\int_\gamma f = F(\gamma(1)) - F(\gamma(0))$ for a smooth path $\gamma : I \to U$. The right hand side of this equation does not involve γ at all and makes sense for arbitrary paths. By Theorem 2.4.2, we know that holomorphic functions have primitives on discs. Our strategy to define the path integral for arbitrary paths is to cover the path by open discs and use primitives for each disc to define the integral piece by piece.

Lemma 3.5.1. *Let $U \subset \mathbb{C}$ be open and $\gamma : I \to U$ continuous. Then there are real numbers $0 = a_0 < \ldots < a_n = 1$ and open discs $D_1, \ldots, D_n \subset U$ such that $\gamma([a_{i-1}, a_i]) \subset D_i$ for $i = 1, \ldots, n$.*



Proof. Since U is open, there is for every $t \in I$ an ϵ_t such that the open disc $D_{\epsilon_t}(\gamma(t))$ with center $\gamma(t)$ is contained in U, and $\mathrm{im}(\gamma)$ is contained in the union of all discs $D_{\epsilon_t}(\gamma(t))$ (for t ranging through all of I). Thus the collection of all $V_t = \gamma^{-1}(D_{\epsilon_t}(\gamma(t))$ (for $t \in I$) is an open cover of I. By Lebesgue's Lemma (Lemma 0.7.3), there is an $n \in \mathbb{N}$ such that for every $i = 1, \ldots, n$, there exists a $t_i \in I$ such that the interval $[\frac{i-1}{n}, \frac{i}{n}]$ is contained in V_{t_i}. Thus the claim of the lemma holds for $D_i = D_{\epsilon_{t_i}}(\gamma(t_i))$ and $a_i = \frac{i}{n}$ for $i = 1, \ldots, n$. $\square$

Consider a holomorphic function $f : U \to \mathbb{C}$ and a path $\gamma : I \to U$. By Lemma 2.5.1, we find $0 = a_0 < \ldots < a_n = 1$ and open discs $D_1, \ldots, D_n \subset U$ such that $\gamma([a_{i-1}, a_i]) \subset D_i$. Let $\gamma_i := \gamma|_{[a_{i-1}, a_i]}$. By Theorem 2.4.2, $f|_{D_i} : D_i \to \mathbb{C}$ has a primitive F_i for every $i = 1, \ldots, n$.

If γ is smooth, then we have

$$\int_\gamma f = \sum_{i=1}^{n} \int_{\gamma_i} f|_{D_i} = \sum_{i=1}^{n} F_i(\gamma(a_i)) - F_i(\gamma(a_{i-1}))$$

by Proposition 2.2.4. The right hand side does not depend on γ being smooth, so we attempt to use this equation as a definition for $\int_\gamma f$ for an arbitrary path γ. The following result verifies the independence from all involved choices.

Definition 3.5.2. Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ a path. Let $0 = a_0 < \ldots < a_n = 1,\ D_1, \ldots, D_n \subset U,\ F_1, \ldots, F_n$ and $\gamma_1, \ldots, \gamma_n$ be as above. Then the *path integral of f along γ* is

$$\int_\gamma f := \sum_{i=1}^{n} F_i(\gamma(a_i)) - F_i(\gamma(a_{i-1})).$$

Proposition 3.5.3. *Let $\gamma : I \to U$, $f : U \to \mathbb{C}$, $a_0, \ldots, a_n$, $D_1, \ldots, D_n$ and $F_1, \ldots, F_n$ as in Definition 2.5.2. Then the complex number*

$$\int_\gamma f = \sum_{i=1}^{n} F_i(\gamma(a_i)) - F_i(\gamma(a_{i-1}))$$

does not depend on the choices of a_i, D_i and F_i.

Proof. We aim to show that $\int_\gamma f$ has the same value for different choices of a_i, D_i and F_i. We consider such alternative choices step by step.

Step 1. Consider fixed $0 = a_0 < \ldots < a_n = 1$ and alternative choices of discs $D'_1, \ldots, D'_n$ and primitives $F'_1, \ldots, F'_n$ of the restriction of f to these discs. Let $\tilde{\gamma}_i : [a_{i-1}, a_i] \to D_i \cap D'_i$ be a smooth path from $z_{i-1} = \gamma(a_{i-1})$ to $z_i = \gamma(a_i)$ (e.g. a linear path). Then by Proposition 2.2.4,

$$F'_i(z_i) - F'_i(z_{i-1}) = \int_{\tilde{\gamma}_i} f = F_i(z_i) - F_i(z_{i-1}).$$

Thus

$$\sum_{i=0}^{n} F'_i(z_i) - F'_i(z_{i-1}) = \sum_{i=0}^{n} F_i(z_i) - F_i(z_{i-1})$$

which shows that the definition of $\int_\gamma f$ is independent from the choices of discs D_i and primitives F_i once the a_i are fixed.

Step 2. Let $k \in \{1, \ldots, n\}$ and refine $0 = a_0 < \ldots < a_n = 1$ by a new element a'_k with $a_{k-1} < a'_k < a_k$. Let $z'_k := \gamma(a'_k)$, $D'_k := D_k$ and $F'_k := F_k$. Then

$$\gamma([a_{k-1}, a'_k]) \subset D'_k \qquad \text{and} \qquad \gamma([a'_k, a_k]) \subset D_k,$$

and F'_k is a primitive of f on D'_k. Therefore

$$\begin{aligned}
\int_{\gamma|_{[a_{k-1},a_k]}} f &= F_k(z_k) - F_k(z_{k-1}) \\
&= \big(F_k(z_k) - F_k(z'_k)\big) + \big(F'_k(z'_k) - F'_k(z_{k-1})\big) \\
&= \int_{\gamma|_{[a_{k-1},a'_k]}} f + \int_{\gamma|_{[a'_k,a_k]}} f.
\end{aligned}$$

In conclusion, $\int f$ does not change under refinements of $0 = a_0 < \ldots < a_n = n$.

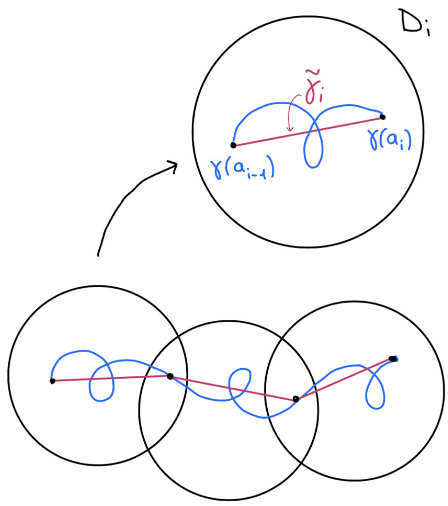
Step 3. Given two sequences $0 = a_0 < \ldots < a_n = 1$ and $0 = b_0 < \ldots < b_m = 1$, we can pass to a common refinement $0 = c_0 < \ldots < c_r = 1$, i.e., $\{a_0, \ldots, a_n, b_0, \ldots, b_m\} \subset \{c_0, \ldots, c_r\}$. Let $F_{a,i}$, $F_{b,i}$ and $F_{c,i}$ be primitives on suitable discs for these three sequences. Since $\int_\gamma f$ is invariant under refinements by step 2, we get

$$\begin{aligned}\sum_{i=1}^{n} F_{a,i}(\gamma(a_i)) - F_{a,i}(\gamma(a_{i-1})) &= \sum_{i=1}^{r} F_{c,i}(\gamma(c_i)) - F_{c,i}(\gamma(c_{i-1})) \\ &= \sum_{i=1}^{m} F_{b,i}(\gamma(b_i)) - F_{b,i}(\gamma(b_{i-1})),\end{aligned}$$

which shows that $\int_\gamma f$ is independent of the choices of a_i, D_i and F_i, as claimed. $\square$

Corollary 3.5.4. *Let $\gamma : I \to U$ be a path, $f : U \to \mathbb{C}$ holomorphic and $0 = a_0 < \ldots < a_n = 1$ such that $\gamma([a_{i-1}, a_i])$ is contained in a disc $D_i \subset U$ for $i = 1, \ldots, n$. Let $\tilde{\gamma}_i$ be the linear path from $\gamma(a_{i-1})$ to $\gamma(a_i)$. Then*

$$\int_\gamma f = \sum_{i=1}^{n} \int_{\tilde{\gamma}_i} f.$$



Proof. Let F_i be a primitive for $f|_{D_i}$. Then either side of the equation in the corollary are equal to $\sum_{i=1}^{n} F_i(\gamma(a_i)) - F_i(\gamma(a_{i-1}))$. $\square$

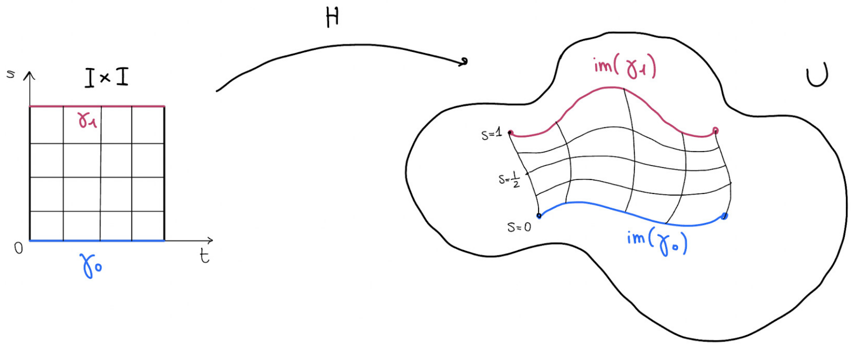
Remark 3.5.5. All properties of Proposition 2.2.4 generalize to any (merely continuous) path; cf. Exercise 2.8.8.

3.6 Homotopies

Definition 3.6.1. Let $U \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : I \to U$ paths. A *homotopy from γ_0 to γ_1 in U* is a continuous map

$$\begin{array}{rccc} H: & I \times I & \longrightarrow & U \\ & (s,t) & \longmapsto & H_s(t) = H(s,t) \end{array}$$

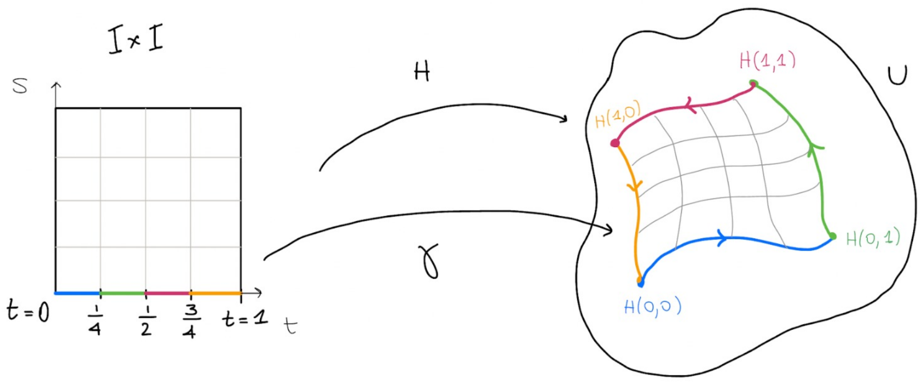
such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$ (as functions from I to U). If there is such a homotopy, we say that *γ_0 and γ_1 are homotopic in U* and write $\gamma_0 \simeq \gamma_1$.



Theorem 3.6.2. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic. Let $H : I \times I \to U$ be continuous and $\gamma : I \to U$ its boundary curve, i.e.,*

$$\gamma(t) = \begin{cases} H(0,4t) & \text{for } 0 \leqslant t < 1/4; \\ H(4t-1,1) & \text{for } 1/4 \leqslant t < 2/4; \\ H(1,3-4t) & \text{for } 2/4 \leqslant t < 3/4; \\ H(4-4t,0) & \text{for } 3/4 \leqslant t \leqslant 1. \end{cases}$$

Then $\int_\gamma f = 0$.



Proof. Since U is open, there is for every $(s,t) \in I \times I$ an $\epsilon(s,t) > 0$ such that the open disc $D_{\epsilon(s,t)}(H(s,t))$ is contained in U. Thus $I \times I$ is equal to the union of all inverse images $V_{s,t} := H^{-1}\big(D_{\epsilon(s,t)}\big(H(s,t)\big)\big)$ for varying $(s,t) \in I \times I$. By Lebesgue's lemma (Lemma 0.7.3), there is an $n > 0$ such that for all $k,l = 1,\dots,n$, there is an $(s,t) \in I \times I$ such that $I_k \times I_l \subset V_{s,t}$ for $I_k := [\frac{k-1}{n}, \frac{k}{n}]$, $I_l := [\frac{l-1}{n}, \frac{l}{n}]$. Thus $H(I_k \times I_l)$ is contained in the open disc $D_{k,l} := D_{\epsilon(s,t)}(H(s,t))$.

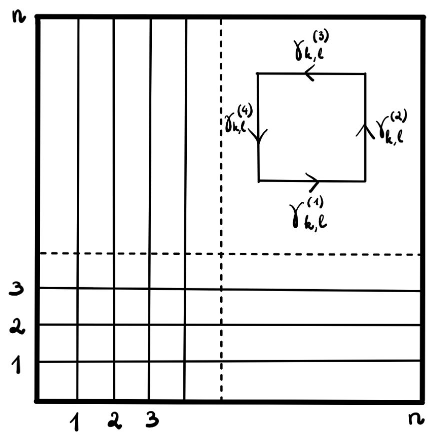


Figure 3.1: The larger square (top right corner) is the enlarged version of one of the smaller squares in the grid (eg. [0,1] × [0,1])

Define for $k,l = 1,\dots,n$ the boundary paths $\gamma^{(i)}_{k,l} : I \to H(I_k \times I_l)$ (for $i = 1,\dots,4$) as

$$\begin{aligned}
\gamma^{(1)}_{k,l}(t) &= H(\tfrac{k-1}{n}, \tfrac{l-1+t}{n}); \\
\gamma^{(2)}_{k,l}(t) &= H(\tfrac{k-1+t}{n}, \tfrac{l}{n}); \\
\gamma^{(3)}_{k,l}(t) &= H(\tfrac{k}{n}, \tfrac{l-t}{n}); \\
\gamma^{(4)}_{k,l}(t) &= H(\tfrac{k-t}{n}, \tfrac{l-1}{n}).
\end{aligned}$$

The concatenation of these paths form the boundary of $H|_{I_k \times I_l}$.

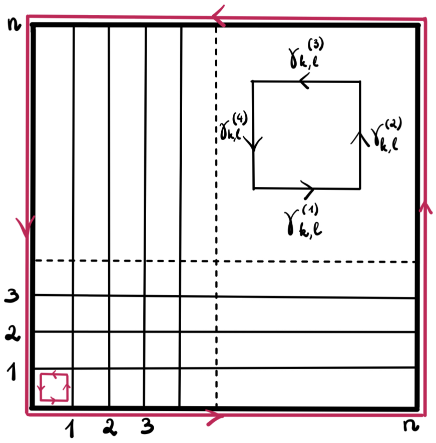


Figure 3.2: Integrating over all the little square becomes the same as integrating over the entire line around the unit square

Since $H(I_k \times I_l)$ is contained in an open disc $D_{k,l}$ and since $f|_{D_{k,l}}$ has a primitive $F_{k,l}$, we have

$$\int_{\gamma_{k,l}^{(1)}} f + \int_{\gamma_{k,l}^{(2)}} f + \int_{\gamma_{k,l}^{(3)}} f + \int_{\gamma_{k,l}^{(4)}} f = F\big(H(\tfrac{k-1}{n}, \tfrac{l-1}{n})\big) - F\big(H(\tfrac{k-1}{n}, \tfrac{l-1}{n})\big) = 0.$$

Together with the observations that

$$\gamma_{k,l+1}^{(1)}(t) = \gamma_{k,l}^{(3)}(1-t) \qquad \text{and} \qquad \gamma_{k,l}^{(2)}(1-t) = \gamma_{k+1,l}^{(4)}(t)$$

for all eligible k and l, and that γ is the concatenation of all paths of the forms

$$\gamma_{0,l}^{(1)}, \qquad \gamma_{k,1}^{(2)}, \qquad \gamma_{1,l}^{(3)}, \qquad \gamma_{k,0}^{(4)}$$

(for $k, l = 1, \ldots, n$), we conclude that

$$\int_\gamma f = \sum_{k,l=1}^{n} \sum_{i=1}^{4} \int_{\gamma_{k,l}^{(i)}} f = 0. \qquad \square$$

3.7 Cauchy's integral theorem

Definition 3.7.1. Let $U \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : I \to U$ paths. A *homotopy from γ_0 to γ_1 in U relative to* $\{0,1\}$ is a homotopy $H : I \times I \to U$ from γ_0 to γ_1 in U such that $H(s,0) = \gamma_0(0) = \gamma_1(0)$ and $H(s,1) = \gamma_0(1) = \gamma_1(1)$ for all $s \in I$. If there is such a homotopy, we say that γ_0 *and* γ_1 *are homotopic in U relative to* $\{0,1\}$ and write $\gamma_0 \simeq \gamma_1$ rel $\{0,1\}$.

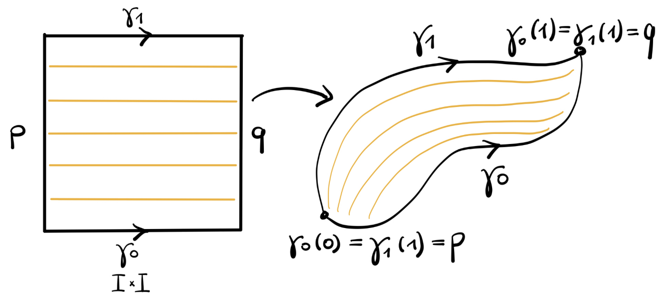


Figure 3.3: Definition 2.7.1

Note that necessarily we must have $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ for γ_0 and γ_1 to be homotopic relative to $\{0,1\}$.

Theorem 3.7.2. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\gamma_0, \gamma_1 : I \to U$ paths that are homotopic in U relative to $\{0,1\}$. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Let $H : I \times I \to U$ be a homotopy from γ_0 to γ_1 relative to $\{0,1\}$. Let $p_0 := \gamma_0(0)$ and $p_1 := \gamma_0(1)$. Let $c_{p_0} : I \to U$ and $c_{p_1} : I \to U$ be the constant paths based on p_0 and p_1, respectively. Then the boundary curve ∂H of H is equal to the concatenation of γ_0, c_{p_1}, γ_1^- and c_{p_0}.

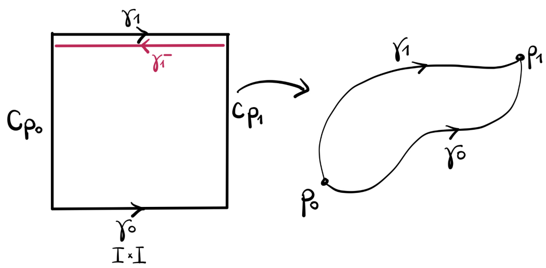


Figure 3.4: Theorem 2.7.2

Thus the claim of the theorem follows from

$$\int_{\gamma_0} f - \int_{\gamma_1} f = \int_{\gamma_0} f + \int_{c_{p_1}} f + \int_{\gamma_1^-} f + \int_{c_{p_0}} f = \int_{\partial H} f = 0,$$

where we use Example 2.2.3.(1) and Proposition 2.2.4.(6) for the first equality and Theorem 2.6.2 for the third equality. □

Definition 3.7.3. Let $U \subset \mathbb{C}$ be open and $\gamma_0, \gamma_1 : I \to U$ closed paths. A *closed homotopy from γ_0 to γ_1 in U* is a homotopy $H : I \times I \to U$ from γ_0 to γ_1 in U such that $H(s,0) = H(s,1)$ for all $s \in I$. If there is such a homotopy, we say that *γ_0 and γ_1 are closed homotopic in U*.

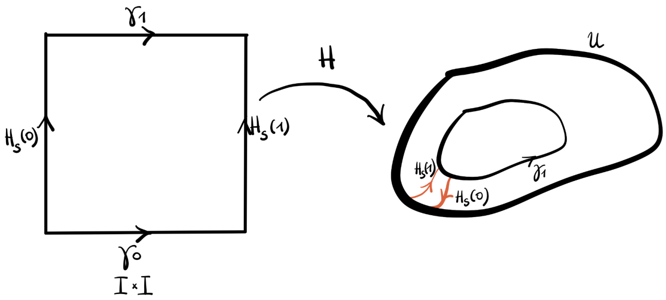


Figure 3.5: Definition 2.7.3

Theorem 3.7.4 (Cauchy's integral theorem, version 1)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\gamma_0, \gamma_1 : I \to U$ closed paths that are closed homotopic. Then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

Proof. Let $H : I \times I \to U$ be a closed homotopy from γ_0 to γ_1 and $\delta(s) = H(s,1)$. Then $\delta(s) = H(s,0)$, and the boundary curve ∂H is equal to the concatenation of γ_0, δ, γ_1^- and δ^-.

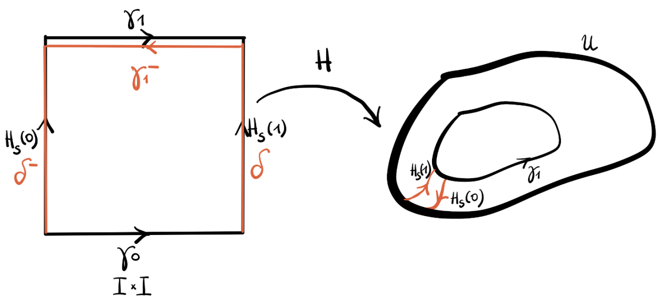


Figure 3.6: Proof of Theorem 2.7.4

Thus the claim of the theorem follows from

$$\int_{\gamma_0} f - \int_{\gamma_1} f = \int_{\gamma_0} f + \int_{\delta} f + \int_{\gamma_1^-} f + \int_{\delta^-} f = \int_{\partial H} f = 0,$$

where we use Proposition 2.2.4.(6) for the first equality and Theorem 2.6.2 for the third equality. □

Definition 3.7.5. Let $U \subset \mathbb{C}$ be open. A closed path $\gamma : I \to U$ is *contractible in U* if it is closed homotopic in U to a constant path.

Theorem 3.7.6 (Cauchy's integral theorem, version 2)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ a contractible closed path. Then $\int_\gamma f = 0$.*

Proof. Since γ is contractible, it is closed homotopic to a constant path $c : I \to U$. By Theorem 2.7.4 and Example 2.2.3.(1), we have $\int_\gamma f = \int_c f = 0$. $\square$

Definition 3.7.7. A *domain* is an open subset $U \subset \mathbb{C}$ that is path-connected. A domain $U \subset \mathbb{C}$ is *simply connected* if every closed path $\gamma : I \to U$ is contractible in U.

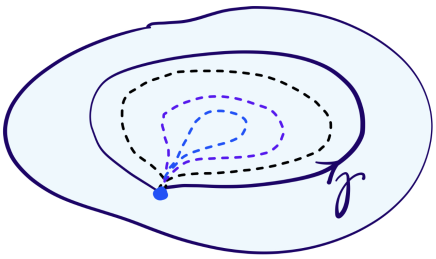


Figure 3.7: Simply connected domain

Theorem 3.7.8 (Cauchy's integral theorem, version 3)**.** *Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ a closed path. Then $\int_\gamma f = 0$.*

Proof. Since U is simply connected, γ is contractible. Thus the result follows at once from Theorem 2.7.6. $\square$

3.8 Exercises

Exercise 3.8.1. Compute the path integral of $f(z) = z$ and of $g(x+iy) = x - iy$ over the closed "triangular" path from 0 to 1 to $\mathbf{i}$ to 0.

Exercise 3.8.2. Let $U \subset \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ holomorphic. Show that the following are equivalent:

(1) f is constant on discs in U.

(2) $f(z) - f(a) \in \mathbb{R}$ if $z, a \in D_r(w)$ for an open disc $D_r(w)$ that is contained in U.

(3) $f'(a) = 0$ for all $a \in U$.

Exercise 3.8.3. Let $\gamma = \{3e^{it} \mid t \in [0, \frac{\pi}{2}]\}$. Find an $N \in \mathbb{R}$ such that

$$\left| \int_\gamma \frac{1}{z^2 + z + 1}\, dz \right| \leqslant N.$$

Exercise 3.8.4. Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a polynomial function. Let γ be a closed path. Show that $\int_\gamma f = 0$.

Exercise 3.8.5. Let $U = \mathbb{C} \setminus \{-1, 1\}$ and $f : U \to \mathbb{C}$ analytic. Assume that $\int_{\gamma_1} f = -1$ and $\int_{\gamma_2} f = 2\pi i$ for $\gamma_i : [0,1] \to U$ with $\gamma_1(t) = 1 + e^{2\pi \mathbf{i} t}$ and $\gamma_2(t) = -1 + e^{2\pi \mathbf{i} t}$. Make an illustration of γ_1 and γ_2, as well as:

$$\gamma_3 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C}, \\ t & \longmapsto & i + 2e^{4\pi \mathbf{i} t}, \end{array} \qquad \gamma_4 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & 2i + 2e^{4\pi \mathbf{i} t}, \end{array}$$

and

$$\gamma_5 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & \begin{cases} -1 + e^{-4\pi \mathbf{i} t} & \text{for } 0 \leqslant t \leqslant 1/2; \\ 1 + e^{4\pi \mathbf{i} t - \mathbf{i}\pi} & \text{for } 1/2 < t \leqslant 1. \end{cases} \end{array}$$

Compute $\int_{\gamma_i} f$ for $i = 3, 4, 5$. Find a function f for which the integrals of this exercise assume the given values.

Exercise 3.8.6. Show that the following open and connected subsets U of $\mathbb{C}$ are simply connected:

$$\mathbb{C}, \qquad \text{an open disc } D_r(Z_0), \qquad \text{a star shaped domain.}$$

Show that the "punctured disc" $D_r^\bullet(z_0) = D_r(z_0) \setminus \{z_0\}$ is *not* simply connected.

Exercise 3.8.7. Let $\gamma : I \to U$ be a closed and contractible path in U with $z_0 = \gamma(0) = \gamma(1)$ and $c_{z_0} : I \to U$ the constant path with $c_{z_0}(t) = z_0$ for all $t \in I$. Show that there is a *closed* homotopy $H : I \times I \to U$ from γ to c_{z_0} in U *relative to* $\{0, 1\}$, i.e. $H_s(0) = H_s(1) = H_1(t) = z_0$ for all $s, t \in I$.

Exercise 3.8.8. Prove that all claims of Proposition 2.2.4 also hold if the path $\gamma : I \to \mathbb{C}$ is (a) piecewise smooth and (b) continuous. This is:

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ piecewise smooth (or merely continuous) and $\gamma : [a,b] \to U$ a path. Then

(1) $\int_\gamma f$ is $\mathbb{C}$-linear in f.

(2) For $a = a_0 \leqslant \ldots \leqslant a_n = b$ and $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$,

$$\int_\gamma f = \sum_{i=1}^n \int_{\gamma_i} f.$$

(3) If $f = F'$ for a holomorphic function $F : U \to \mathbb{C}$, then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

(4) Let $\alpha : [c,d] \to [a,b]$ be continuously differentiable with $\alpha(c) = a$ and $\alpha(d) = b$. Then

$$\int_{\gamma \circ \alpha} f = \int_\gamma f.$$

(5)
$$\left| \int_{\gamma} f \right| \quad \leqslant \quad \ell(\gamma) \cdot \max\{|f(z)| \mid z \in \operatorname{im}(\gamma)\}.$$

(6) Let $\gamma^- : [a,b] \to U$ be defined by $\gamma^-(t) = \gamma(a+b-t)$. Then
$$\int_{\gamma^-} f \;=\; -\int_{\gamma} f.$$

Chapter 4

Analytic functions

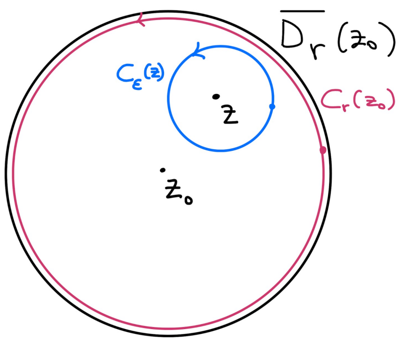
4.1 Cauchy's integral formula

Recall that $C_r(z) : I \to \mathbb{C}$ is the circle of radius r around z, given by $C_r(z)(t) = z + re^{2\pi \mathbf{i} \cdot t}$. We denote by $\overline{D}_r(z) = \{w \in \mathbb{C} \mid |w - z| \leqslant r\}$ the *closed disc of radius r with center z*, which is the closure of the open disc $D_r(z)$.

Theorem 4.1.1 (Cauchy's integral formula)**.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic. Consider a closed disc $\overline{D}_r(z_0) \subset U$ for some $r > 0$ and $z_0 \in U$ and $z \in D_r(z_0)$. Then*

$$f(z) = \frac{1}{2\pi \mathbf{i}} \int\limits_{C_r(z_0)} \frac{f(w)}{w - z}\, dw.$$

Proof. The function $\frac{f(w)}{w-z}$ is holomorphic in $w \in U \setminus \{z\}$. For $0 < \epsilon < r - |z - z_0|$, the circular path $C_\epsilon(z)$ is contained in $U \setminus \{z\}$ and closed homotopic to $C_r(z_0)$ in $U \setminus \{z\}$.



By Cauchy's integral theorem (Theorem 2.7.4), we have

$$\int\limits_{C_r(z_0)} \frac{f(w)}{w - z}\, dw = \int\limits_{C_\epsilon(z)} \frac{f(w)}{w - z}\, dw = \int\limits_{C_\epsilon(z)} \frac{f(w) - f(z)}{w - z}\, dw + \int\limits_{C_\epsilon(z)} \frac{f(z)}{w - z}\, dw.$$

Since $\lim_{w\to z} \frac{f(w)-f(z)}{w-z} = f'(w)$, we have

$$\left| \int\limits_{C_\epsilon(z)} \frac{f(w)-f(z)}{w-z}\, dw \right| \leqslant 2\pi\epsilon \cdot \max\left\{ \left| \frac{f(w)-f(z)}{w-z} \right| \,\middle|\, w \in \mathrm{im}(C_\epsilon(z)) \right\} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

and since the integral is independent of ϵ, we conclude that $\int_{C_\epsilon(z)} \frac{f(w)-f(z)}{w-z}\, dw = 0$. Thus

$$\int\limits_{C_r(z_0)} \frac{f(w)}{w-z}\, dw \;=\; \int\limits_{C_\epsilon(z)} \frac{f(z)}{w-z}\, dw \;=\; f(z)\cdot \int\limits_{C_\epsilon(0)} \frac{1}{u}\, du \;=\; 2\pi\mathbf{i}\cdot f(z),$$

where we use $u = w-z$ in the second equality (note that $\frac{\partial u}{\partial w} = 1$) and Example 2.2.3.(3) in the third equality. Dividing both sides by $2\pi\mathbf{i}$ yields the result. □

Example 4.1.2. We can use Cauchy's integral formula to compute path integrals, such as

$$\int\limits_{C_2(0)} \frac{e^w}{w-1}\, dw \;=\; \int\limits_{C_2(0)} \frac{f(w)}{w-z}\, dw \;=\; 2\pi\mathbf{i}\cdot f(z) \;=\; 2\pi\mathbf{i}\cdot e,$$

where we use $f(w) = e^w$ and $z = 1$.

Corollary 4.1.3 (mean value principle)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $\overline{D}_r(z) \subset U$ for some $r > 0$ and $z \in U$. Then*

$$f(z) \;=\; \frac{1}{2\pi} \cdot \int_0^{2\pi} f(z + re^{it})\, dt.$$

Proof. By Cauchy's integral formula (Theorem 3.1.1),

$$\begin{aligned} f(z) \;&=\; \frac{1}{2\pi\mathbf{i}} \cdot \int\limits_{C_r(z)} \frac{f(w)}{w-z}\, dw \\ &=\; \frac{1}{2\pi\mathbf{i}} \cdot \int_0^{2\pi} \frac{f(z+re^{it})}{re^{it}} \cdot \mathbf{i}re^{it}\, dt \\ &=\; \frac{1}{2\pi} \cdot \int_0^{2\pi} f(z+re^{it})\, dt, \end{aligned}$$

where we use Proposition 2.2.4.(4) with respect to the bijection $\alpha : [0, 2\pi] \to [0, 1]$ with $\alpha(t) = \frac{t}{2\pi}$ and substitute $w = z + re^{it}$ in the second equality (note that $\frac{\partial w}{\partial t} = \mathbf{i}re^{it}$). □

4.2 The maximum modulus principle

Lemma 4.2.1. *Let $f : D_r(z) \to \mathbb{C}$ be holomorphic where $r > 0$ and $z \in \mathbb{C}$. If $|f(w)| \leqslant |f(z)|$ for all $w \in D_r(z)$, then $|f(w)|$ is constant (and equal to $|f(z)|$) for $w \in D_r(z)$.*

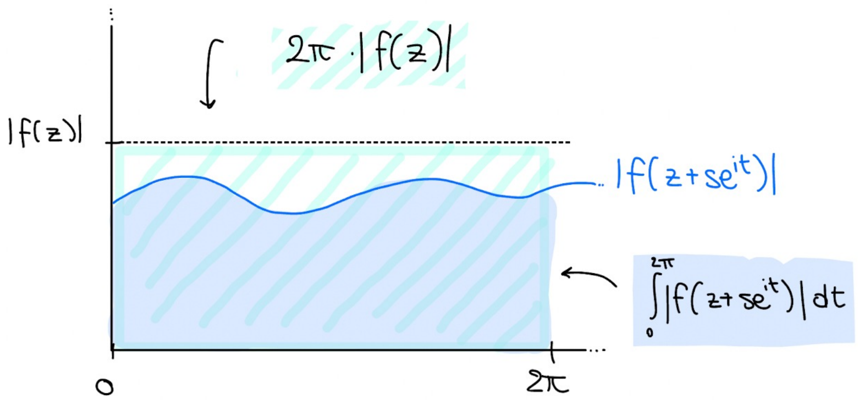
Proof. By the mean value principle (Corollary 3.1.3) and Proposition 2.2.4.(5), we have for any $0 < s < r$ that

$$\begin{aligned}|f(z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(z+se^{it})\,dt\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|f(z+se^{it})\right|dt \\ &\leqslant \frac{1}{2\pi}\cdot 2\pi\cdot\max\left\{|f(z+se^{it})|\,\middle|\,0\leqslant t\leqslant 2\pi\right\} \leqslant |f(z)|.\end{aligned}$$

We conclude that all inequalities are equalities and that

$$\int_0^{2\pi}\left|f(z+se^{it})\right|dt = 2\pi\cdot|f(z)|.$$

Since $|f(z+se^{it})|$ is continuous in t and has values in $[0,|f(z)|]$, the previous equality implies that $|f(z+se^{it})| = |f(z)|$ for all $t \in [0,2\pi]$ (cf. Exercise 3.7.1).



Since every $w \in D_r(z)\setminus\{z\}$ is of the form $w = z+se^{it}$ for some $s \in (0,r)$ and $t \in [0,2\pi]$, the result follows. $\square$

Theorem 4.2.2 (maximum modulus principle)**.** *Let $K \subset \mathbb{C}$ be compact and $f : K \to \mathbb{C}$ a continuous function whose restriction to the interior of K is holomorphic. Then the function*

$$\begin{aligned}|f|:\ K &\longrightarrow \mathbb{R}\\ z &\longmapsto |f(z)|\end{aligned}$$

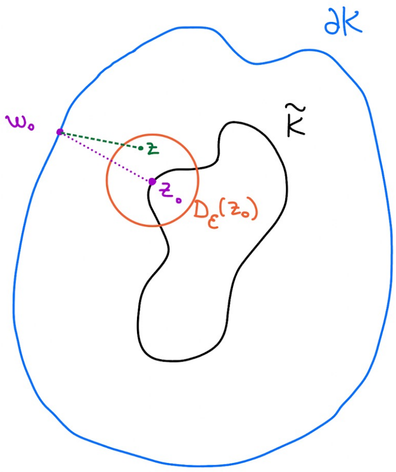
assumes its maximum on the boundary of K, i.e., there is a w in the boundary of K such that $|f(z)| \leqslant |f(w)|$ for all z in K.

Proof. As the composition of two continuous functions, f and $|\cdot|$, the function $|f| : K \to \mathbb{R}$ is continuous. Since K is compact, $|f|$ assumes its maximum on K, i.e., there is a $z_0 \in K$ such that $|f(z)| \leqslant |f(z_0)| = M$ for all $z \in K$; cf. Lemma 0.7.2.

Let ∂K be the boundary of K and $\widetilde{K} := |f|^{-1}(M)$. Both subsets are closed since $K \setminus \mathring{K}$ and $\{M\}$ are closed. They are bounded as subsets of K, and therefore compact. Thus the Cartesian product $\partial K \times \widetilde{K}$ is compact and the function

$$
\begin{array}{rccc}
d: & \partial K \times \widetilde{K} & \longrightarrow & \mathbb{R} \\
 & (w,z) & \longmapsto & |w-z|
\end{array}
$$

assumes its minimum in some point $(w_0, \tilde{z}_0) \in \partial K \times \widetilde{K}$ (again, cf. Lemma 0.7.2).



If $|w_0 - \tilde{z}_0| > 0$, then we have for all sufficiently small $\epsilon > 0$ and $z \in D_\epsilon(z_0) \subset K$ that $|f(z)| \leqslant |f(z_0)| = M$. By Lemma 3.2.1, $|f(z)| = |f(z_0)|$ for all $D_\epsilon(z_0)$ and thus $|w_0 - z| < |w_0 - \tilde{z}_0|$ for some $z \in D_\epsilon(\tilde{z}_0)$, which is a contradiction since we assumed that the minimum of d was attained at $(w_0, \tilde{z}_0)$.

We conclude that $|w_0 - \tilde{z}_0| = 0$, i.e., $|f|$ assumes its maximum in $\tilde{z}_0 = w_0 \in \partial K$. □

4.3 Analytic functions

Definition 4.3.1. Let $z_0 \in \mathbb{C}$. A *power series around* z_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

with $a_0, a_1, \ldots \in \mathbb{C}$.

Theorem 4.3.2 (Cauchy-Hadamard, version 2)**.** *Let $\sum a_n (z-z_0)^n$ be a power series around $z_0 \in \mathbb{C}$ and*

$$r = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

its radius of convergence. *Then* $\sum a_n(z-z_0)^n$ *converges absolutely for all* $z \in D_r(z_0)$ *and defines a holomorphic function*

$$\begin{array}{rccc} f: & D_r(z_0) & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum a_n(z-z_0)^n \end{array}$$

with derivative

$$f'(z) \;=\; \sum_{n=0}^{\infty} (n+1)\cdot a_{n+1}\cdot (z-z_0)^n$$

for $z \in D_r(z_0)$. *In particular, the radius of convergence of* $\sum (n+1)a_{n+1}(z-z_0)^n$ *is* r.

Proof. This follows from Theorem 1.3.4 and Theorem 1.3.7 if we substitute $z-z_0$ by w. □

In the following, we briefly write that

$$\begin{array}{rccc} f: & D_r(z_0) & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum a_n(z-z_0)^n \end{array}$$

is holomorphic, by which we mean that $\sum a_n(z-z_0)^n$ converges for all $z \in D_r(z_0)$. We denote the n-th derivative of a sufficiently complex differentiable function $f: U \to \mathbb{C}$ in z_0 by

$$f^{(n)}(z_0) \;:=\; \frac{d^n}{dz^n} f(z_0).$$

Corollary 4.3.3. *Let*

$$\begin{array}{rccc} f: & D_r(z_0) & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum a_n(z-z_0)^n \end{array}$$

be holomorphic. Then f *is arbitrarily often complex differentiable (in all* z *in* $D_r(z_0)$*) and*

$$a_n \;=\; \frac{1}{n!}\, f^{(n)}(z_0).$$

Proof. A repeated application of Theorem 3.3.2 yields

$$f^{(k)}(z_0) \;=\; \sum_{n=k}^{\infty} n\cdot(n-1)\cdots(n-k+1)\cdot a_n\cdot(z_0-z_0)^{n-k} \;=\; k!\cdot a_k$$

where we apply the usual convention that $0^{n-k}=0$ for $n-k>0$ and $0^0=1$. This shows that f is arbitrarily often complex differentiable. Dividing both sides by $k!$ proves the second part of the claim. □

Definition 4.3.4. Let $U \subset \mathbb{C}$ be open. An *analytic function in* U is a function $f: U \to \mathbb{C}$ that is arbitrarily often complex differentiable such that for every $z_0 \in U$, there is an $r>0$ such that $D_r(z_0) \subset U$ and

$$f(z) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n$$

for every $z \in D_r(z_0)$. In particular, this assumes that the right hand side of this equation converges, which is called the *Taylor expansion of* f *at* z_0.

Example 4.3.5. Let $U = \mathbb{C} \setminus \{1\}$ and $f : U \to \mathbb{C}$ be given by $f(z) = \frac{1}{1-z}$. Since $f(z)$ is the multiplicative inverse of $(1-z)$ and since the Cauchy product $(1-z) \cdot \sum z^n$ equals 1, we conclude that the Taylor expansion of f at $z_0 = 0$ is the geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

We know already that it converges on the unit disc $D_1(0)$. In particular this means f restricted to $D_1(0)$ is analytic.

4.4 Holomorphic functions are analytic

In this section, we show that holomorphic functions are analytic. In a critical step in the proof, we need to exchange a limit with an integral, which requires the uniform convergence of a sequence of functions. We provide the necessary background on this before we turn to the central theorem of this section.

Definition 4.4.1. Let $A \subset \mathbb{C}$ be a subset, $\{f_n : A \to \mathbb{C}\}_{n \in \mathbb{N}}$ a sequence of functions and $f : A \to \mathbb{C}$ a function. The sequence $\{f_n\}$ *converges uniformly to* f if for every $\epsilon > 0$, there is an $N > 0$ such that for all $z \in A$ and $n \geqslant N$,

$$\left| f(z) - f_n(z) \right| < \epsilon.$$

We write $f_n \rightrightarrows f$ if $\{f_n\}$ converges uniformly to f.

Lemma 4.4.2. *Let $\gamma : [0,1] \to \mathbb{C}$ be a path with image Γ and $\{f_n : \Gamma \to \mathbb{C}\}_{n \in \mathbb{N}}$ a sequence of continuous functions that converges uniformly to a continuous function $f : \Gamma \to \mathbb{C}$. Then*

$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. Since $f_n \rightrightarrows f$, there is for every $\epsilon > 0$ an $N > 0$ such that $|f_n(z) - f(z)| < \epsilon$ for all $z \in \Gamma$ and all $n \geqslant N$. Using Proposition 2.2.4.(5), we derive

$$\left| \int_\gamma f - \int_\gamma f_n \right| \leqslant \int_\gamma \left| f(z) - f_n(z) \right| dz < \ell(\gamma) \cdot \epsilon.$$

Since $\ell(\gamma)$ is fixed, this implies that $\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f$, as desired. □

Theorem 4.4.3. *Let $\gamma : I \to \mathbb{C}$ be a path with image Γ and $h : \Gamma \to \mathbb{C}$ continuous. Then the function $f : \mathbb{C} \setminus \Gamma \to \mathbb{C}$ defined by*

$$f(z) = \int_\gamma \frac{h(w)}{w - z} \, dw$$

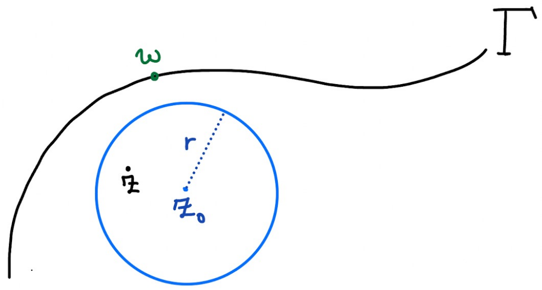
is analytic with n-th derivative

$$f^{(n)}(z) \;=\; n! \cdot \int\limits_{\gamma} \frac{h(w)}{(w-z)^{n+1}}\, dw$$

for $z \in \mathbb{C} \setminus \Gamma$.

Proof. Since $U := \mathbb{C} \setminus \Gamma$ is open, there is for every $z_0 \in U$ an $r > 0$ such that $\overline{D}_r(z_0) \subset U$. Fix such z_0 and $r > 0$. For $w \in \Gamma$ and $z \in D_r(z_0)$, we define $q := \frac{z-z_0}{w-z_0}$, which has absolute value

$$|q| \;=\; \frac{|z-z_0|}{|w-z_0|} \;<\; \frac{r}{r} \;=\; 1.$$



Thus the geometric series $\sum q^n$ converges to $\frac{1}{1-q}$, which gives us

$$\frac{1}{w-z} \;=\; \frac{1}{(w-z_0)-(z-z_0)} \;=\; \frac{1}{w-z_0} \cdot \frac{1}{1-q} \;=\; \frac{1}{w-z_0} \cdot (1+q+q^2+\cdots).$$

Next we define the functions

$$S_N(w) \;:=\; \frac{h(w)}{w-z_0} \cdot \sum_{n=0}^{N} q^n \qquad \text{and} \qquad S_\infty(w) \;:=\; \frac{h(w)}{w-z_0} \cdot \sum_{n=0}^{\infty} q^n,$$

where $w \in \Gamma$, $N \in \mathbb{N}$ and where we consider z_0 and z as fixed constants. Let $M := \max\{|h(w)| \mid w \in \Gamma\}$ and $q_0 := \frac{|z-z_0|}{r}$, which is a real number with $|q| \leqslant q_0 < 1$. Thus

$$\begin{aligned} \big|S_\infty(w) - S_N(w)\big| \;&=\; \left| \frac{h(w)}{w-z_0} \cdot \sum_{n=N+1}^{\infty} q^n \right| \\ &\leqslant\; \frac{|h(w)|}{|w-z_0|} \cdot \sum_{n=N+1}^{\infty} |q|^n \\ &<\; \frac{M}{r} \cdot \sum_{n=N+1}^{\infty} \left(\frac{|z-z_0|}{r} \right)^n \\ &=\; \frac{M}{r} \cdot \frac{q_0^{N+1}}{1-q_0} \quad \xrightarrow[N\to\infty]{} \quad 0 \end{aligned}$$

where the first inequality is the triangle inequality and the second inequality follows from $|q| < q_0$. Since the the last line of these equations does not depend on w anymore, the convergence is uniform in w, i.e., $S_N(w) \rightrightarrows S_\infty(w)$. Therefore

$$f(z) \;=\; \int\limits_\gamma \frac{h(w)}{w-z}\,dw \;=\; \int\limits_\gamma \left(\sum_{n=0}^{\infty} \frac{h(w)}{w-z_0} \cdot q^n \right) dw$$
$$= \sum_{n=0}^{\infty} \left(\int\limits_\gamma \frac{h(w)}{(w-z_0)^{n+1}}\,dw \right) \cdot (z-z_0)^n$$

where we apply Lemma 3.4.2 in order to exchange the infinite sum with the integral in the third equality. The last expression is a power series around z_0 with coefficients

$$a_n \;:=\; \int\limits_\gamma \frac{h(w)}{(w-z_0)^{n+1}}\,dw;$$

since it equals $f(z)$, this power series converges for $z \in D_r(z_0)$. This shows that f is analytic (recall that z_0 was chosen arbitrarily). By Corollary 3.3.3, the n-th derivative of f is

$$f^{(n)}(z_0) \;=\; n! \cdot a_n \;=\; n! \cdot \int\limits_\gamma \frac{h(w)}{(w-z_0)^{n+1}}\,dw,$$

as claimed. □

Theorem 4.4.4. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic. Then f is analytic and*

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi \mathbf{i}} \cdot \int\limits_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

whenever $z \in D_r(z_0)$ and $\overline{D}_r(z_0) \subset U$.

Proof. By Cauchy's integral formula (Theorem 3.1.1), we have

$$f(z) \;=\; \frac{1}{2\pi \mathbf{i}} \cdot \int\limits_{C_r(z_0)} \frac{f(w)}{w-z}\,dw$$

if $z \in D_r(z_0)$ and $\overline{D}_r(z_0) \subset U$. Theorem 3.4.3 applied to the function $h(w) := \frac{f(w)}{2\pi \mathbf{i}}$ verifies that f is analytic and that

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi \mathbf{i}} \cdot \int\limits_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw,$$

as claimed. □

Corollary 4.4.5 (Taylor expansion)**.** *Let $U \subset \mathbb{C}$ be open, $z_0 \in U$ and $r > 0$ such that $D_r(z_0) \subset U$. Let $f : U \to \mathbb{C}$ be holomorphic and set $a_n := \frac{1}{n!} f^{(n)}(z_0)$. Then*

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D_r(z_0)$. In particular, $\sum a_n (z-z_0)^n$ converges for all $z \in D_r(z_0)$.

Proof. Let $z \in D_r(z_0)$. Then there is an $s > 0$ such that $z \in D_s(z_0)$ and $\overline{D}_s(z_0) \subset U$. By Cauchy's integral formula (Theorem 3.1.1), we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathbf{i}} \cdot \int_{C_s(z_0)} \frac{f(w)}{w - z}\, dw \\ &= \sum_{n=0}^{\infty} \left(\int_{C_s(z_0)} \frac{f(w)}{2\pi \mathbf{i} \cdot (w - z_0)^{n+1}}\, dw \right) \cdot (z - z_0)^n \\ &= \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n \end{aligned}$$

where we use the uniform convergence of the geometric sequence in $q = \frac{z - z_0}{w - z_0}$ (as established in the proof of Theorem 3.4.3) in the second equality and the identification

$$\int_{C_s(z_0)} \frac{f(w)}{2\pi \mathbf{i} \cdot (w - z_0)^{n+1}}\, dw \;=\; \frac{1}{n!} f^{(n)}(z_0) \;=\; a_n$$

(from Theorem 3.4.4) in the third equality. This establishes the desired equality $f(z) = \sum a_n (z - z_0)^n$ and shows that the power series converges for all $z \in D_r(z_0)$. $\square$

4.5 Liouville's theorem

Definition 4.5.1. Let $U \subset \mathbb{C}$. A function $f : U \to \mathbb{C}$ is *bounded* if there exists a *bound* $M \in \mathbb{R}$ such that $|f(z)| \leqslant M$ for all $z \in U$.

Theorem 4.5.2 (Liouville)**.** *Every bounded entire function is constant.*

Proof. Let $f : \mathbb{C} \to \mathbb{C}$ be bounded and entire with $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. By Corollary 3.4.5,

$$f(z) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, z^n$$

for all $z \in \mathbb{C}$. By Theorem 3.4.4 and Proposition 2.2.4.(5), we have for all $r > 0$ and $n \geqslant 1$ that

$$\left| \frac{f^{(n)}(0)}{n!} \right| \;=\; \left| \frac{1}{2\pi \mathbf{i}} \cdot \int_{C_r(0)} \frac{f(w)}{(w - 0)^{n+1}}\, dw \right|$$

$$\leqslant \frac{1}{2\pi} \cdot \ell(C_r(0)) \cdot \frac{M}{r^{n+1}} = \frac{M}{r^n} \quad \xrightarrow[r\to\infty]{} \quad 0$$

where we use that $|w| = r$ since $w \in \mathrm{im}(C_r(0))$ for the first inequality. We use $\ell(C_r(0)) = 2\pi \cdot r$ in the last equality. This shows that

$$f(z) = f(0) + 0 \cdot z + 0 \cdot z^2 + \cdots = f(0),$$

i.e. f is constant. □

We mention the following complementary result without proof.

Theorem 4.5.3 (Little Picard theorem)**.** *The image of a non-constant entire function is either $\mathbb{C}$ or $\mathbb{C} \setminus \{b\}$ for some $b \in \mathbb{C}$.*

4.6 The fundamental theorem of algebra

Theorem 4.6.1 (Fundamental Theorem of Algebra)**.** *Let $f = \sum_{n=0}^{d} a_n z^n$ be a complex polynomial of degree d, i.e., $a_0, \dots, a_d \in \mathbb{C}$ and $a_d \neq 0$. If $d > 0$, then f has a* zero *z_0, i.e., $f(z_0) = 0$.*

Proof. Assume that $f(z) \neq 0$ for all $z \in \mathbb{C}$. Then $\frac{1}{f} : \mathbb{C} \to \mathbb{C}$ is entire. Then there is an $r > 0$ such that for $|z| > r$, we have

$$|f(z)| \geqslant \frac{1}{2} \cdot |a_d| \cdot |z|^d > |a_0| = |f(0)|.$$

Thus $|\frac{1}{f}(z)| < |\frac{1}{f}(0)|$ for $|z| > r$. Since $\overline{D}_r(0)$ is compact, the set $\{\frac{1}{f}(z) \mid |z| \leqslant r\}$ is compact and thus bounded. In conclusion, $\frac{1}{f}$ is bounded and therefore constant by Liouville's theorem (Theorem 3.5.2). This shows that $f(z) = a_0$ has degree $d = 0$. It follows by contraposition that $d > 0$ implies that f has a zero. In conclusion, every polynomial of positive degree has a zero. □

Corollary 4.6.2. *Let $f = \sum_{n=0}^{d} a_n z^n$ be a complex polynomial of degree d. Then there are $c_1, \dots, c_d \in \mathbb{C}$ such that*

$$f = a_d \cdot \prod_{i=1}^{d} (z - c_i).$$

Proof. The proof is left as an exercise; cf. Exercise 3.7.6. □

4.7 Exercises

Exercise 4.7.1. Let $f : [a,b] \to [0,c]$ a continuous function where $a < b$ and $0 < c$. Assume that $\int_a^b f(t)\, dt = (b-a) \cdot c$. Show that $f(t) = c$ for all $t \in [a,b]$.

Insert an illustration

Exercise 4.7.2.
Show that polynomials are analytic functions.

Exercise 4.7.3.
Compute the Taylor expansions of $f(z) = 1/(1-z)$ at $z = 1+i$ and $z = 1-i$.

Exercise 4.7.4.
Let $A \subset \mathbb{C}$ be a subset. Let $\{f_n : A \to \mathbb{C}\}_{n\in\mathbb{N}}$ be a sequence of continuous functions that converges *pointwise* on A to $f : A \to \mathbb{C}$ (i.e. for all $\epsilon > 0$ and $z \in A$ there is an $N > 0$ such that $|f(z) - f_n(z)| < \epsilon$ for all $n \geqslant N$).

(1) Show that f is uniquely determined as the (pointwise) limit of $\{f_n\}$.

(2) Show that f is continuous if f_n converges uniformly to f.

(3) Give an example of continuous functions $f_n : A \to \mathbb{C}$ that converge pointwise to a *non*-continuous function $f : A \to \mathbb{C}$.

Exercise 4.7.5.
Compute the Taylor expansion $\sum a_n(z-1)^n$ of $g(z) = e^z - e$ at $z = 1$ and show that $g(z) = \sum a_n(z-1)^n$ for all $z \in \mathbb{C}$. Show that $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ with $f(z) = g(z)/(1-z)$ extends to a holomorphic function $f : \mathbb{C} \to \mathbb{C}$. Conclude that

$$f(z) \;=\; \sum_{n=0}^{\infty} \left[\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} - e \right] z^n$$

for all $z \in \mathbb{C}$ and, in particular, that the right hand side converges for all $z \in \mathbb{C}$.

Exercise 4.7.6. Let $f = \sum a_n z^n$ be a polynomial of of degree d.

(1) Let $c \in \mathbb{C}$ be a zero of f. Show that there is a polynomial $g = \sum b_n z^n$ of degree $d-1$ such that $f(z) = (z-c) \cdot g(z)$ for all $z \in \mathbb{C}$.

Hint: This follows because the Taylor expansion of f at c is of degree d (why?) and has constant coefficient 0 (why?).

(2) Conclude that there are $c_1, \ldots, c_d \in \mathbb{C}$ such that

$$f(z) \;=\; a_d \cdot \prod_{i=1}^{d} (z - c_i)$$

for all $z \in \mathbb{C}$. Show further that $c_1, \ldots, c_d$ are uniquely determined up to a permutation of indices.

Chapter 5

Residues

Motivation. The central result of this chapter is Cauchy's *residue theorem*, which identifies the path integral of a holomorphic function along a path with an easily computable expression. Under simplified hypotheses, the residue theorem reads as the following.

Let $U \subset \mathbb{C}$ be open, $\gamma : I \to U$ a path and $f : U \to \mathbb{C}$ be holomorphic. Assume that there is an $r > 0$ such that $\operatorname{im}\gamma \subset D_r(0)$ and such that $D_r(0) \setminus U$ is finite. Then

$$\int_\gamma f \;=\; 2\pi\mathbf{i} \sum_{c \in D_r(0)\setminus U} W(\gamma, c) \cdot \operatorname{Res}_c(f)$$

where the elements of $D_r(0) \setminus U$ are *isolated singularities of* f, $W(\gamma, c)$ is the *winding number of* γ *around* c (which counts the number of times that γ circles around c in counter-clockwise direction) and $\operatorname{Res}_c(f)$ is the *residue of* f *at* a, which equals the coefficient a_{-1} of the *Laurent expansion* $f = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$ *of* f *about* c.

We will introduce all of these notions in the upcoming sections.

5.1 Singularities

Motivation. We study the behaviour of the functions

$$f_1(z) \;=\; \frac{e^z - 1}{z}, \qquad f_2(z) \;=\; \frac{1}{z}, \qquad f_3(z) \;=\; e^{\frac{1}{z}}$$

when $z \to 0$:

(1) We have

$$f_1(z) \;=\; \frac{1}{z} \cdot \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 \right) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \;\xrightarrow[z \to 0]{}\; 1$$

which means that f_1 extends to holomorphic function $f_1 : \mathbb{C} \to \mathbb{C}$ with value $f_1(0) = 1$. (*removable singularity*)

(2) We have

$$|f_2(z)| = \frac{1}{|z|} \underset{z\to 0}{\longrightarrow} \infty.$$

(*pole*)

(3)

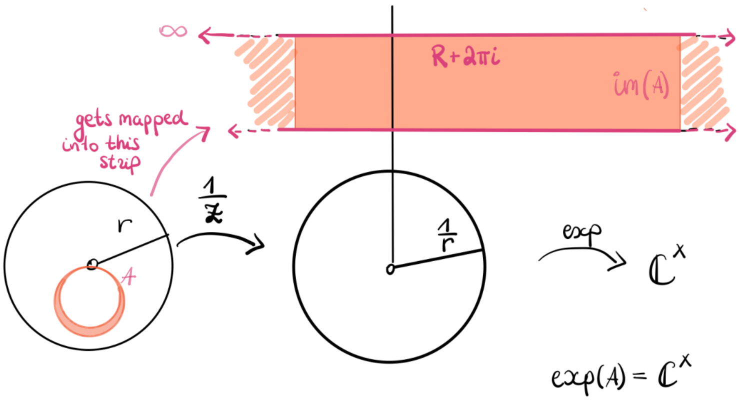


Figure 5.1: Essential singularity where a point passing through a pole at zero makes the limit be mapped into a horizontal strip of same color

Thus $f_3(D_r \setminus \{0\}) = \mathbb{C}^\times$ for all $r > 0$. (*essential singularity*)

Definition 5.1.1. Let $a \in \mathbb{C}$ and $r > 0$. The *punctured disc of radius r around a* is

$$D_r^\bullet(a) := \{z \in \mathbb{C} \mid 0 < |z-a| < r\} = D_r(a) \setminus \{a\}.$$

Let $U \subset \mathbb{C}$ be open with $D_r^\bullet \subset U$ and $f : U \to \mathbb{C}$ a nontrivial holomorphic function (i.e. f is not constant zero). The *order of f in a* is

$$\operatorname{ord}_a(f) := \sup\left\{ m \in \mathbb{Z} \;\middle|\; \begin{array}{c}\text{there is a holomorphic } h : D_r(a) \to \mathbb{C} \text{ such} \\ \text{that } f(z) = (z-a)^m h(z) \text{ for all } z \in D_r^\bullet(a)\end{array} \right\},$$

which is, by definition, equal to $-\infty$ if the set is empty.

We call a a *zero (of order n) of f* if $a \in U$ and if $n = \operatorname{ord}_a(f)$ is positive.

We call a an *(isolated) singularity of f* if $a \notin U$. In this case, a is

- a *removable singularity* if $\operatorname{ord}_a(f) \geqslant 0$;
- a *pole (of order n)* if $-\infty < \operatorname{ord}_a(f) = -n < 0$;
- an *essential singularity* if $\operatorname{ord}_a(f) = -\infty$.

We say that f *extends analytically to a* if there is a holomorphic function $h : D_r(a) \to \mathbb{C}$ such that $h(z) = f(z)$ for all $z \in D_r^\bullet(a)$.

More generally, let $U \subset V \subset \mathbb{C}$ be open subsets of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. We say that f *extends analytically to V* if there is a holomorphic function $h : V \to \mathbb{C}$ such that $f = h|_U$.

Remark 5.1.2.

(1) If $f(z) = (z-a)^m h(z)$ for $z \in D_r^\bullet(a)$, $m \in \mathbb{Z}$ and holomorphic $h : D_r(a) \to \mathbb{C}$ with $h(a) \neq 0$, then $\mathrm{ord}_a(f) = m$.

(2) The order satisfies

$$\begin{aligned} \mathrm{ord}_a(f \cdot g) &= \mathrm{ord}_a(f) + \mathrm{ord}_a(g); \\ \mathrm{ord}_a\left(\frac{1}{f}\right) &= -\mathrm{ord}_a(f); \\ \mathrm{ord}_a(f+g) &\geqslant \min\{\mathrm{ord}_a(f),\ \mathrm{ord}_a(g)\}; \end{aligned}$$

cf. Exercise 4.8.1.

(3) If $a \in U$, then $\mathrm{ord}_a(f) \geqslant 0$ and $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for $z \in D_r(a)$. In this case, $\mathrm{ord}_a(f) > 0$ if and only if $f(a) = 0$.

(4) If $a \notin U$, then a is a removable singularity if and only if f extends analytically to a.

Example 5.1.3.

(1) Let $b \in \mathbb{C}^\times$. Then

$$\begin{array}{rccc} f: & \mathbb{C}^\times & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & b \end{array}$$

has $\mathrm{ord}_0(f) = 0$ and $z = 0$ is a removable singularity.

(2) The function $f(z) = z^n$ has $\mathrm{ord}_0(f) = n$. More generally, a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (with positive radius of convergence) has order $\mathrm{ord}_0(f) = \min\{i \in \mathbb{Z} \mid a_i \neq 0\}$.

(3) Let $u_1, \dots, u_d, v_1, \dots, v_e \in \mathbb{C}$ be pairwise distinct and $\delta_1, \dots, \delta_d, \epsilon_1, \dots, \epsilon_e \in \mathbb{Z}_{\geqslant 1}$. Then the rational function

$$f(z) = \frac{\prod_{i=1}^{d} (z-u_i)^{\delta_i}}{\prod_{i=1}^{e} (z-v_i)^{\epsilon_i}}$$

has zeros u_i of order $\delta_i = \mathrm{ord}_{u_i}(f)$ and poles v_i of order $\epsilon_i = -\mathrm{ord}_{v_i}(f)$.

Theorem 5.1.4 (Riemann's theorem on removable singularities)**.** *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic, $a \in \mathbb{C}$ a singularity of f and $D_r^\bullet(a) \subset U$. If $f|_{D_r^\bullet(a)}$ is bounded, then a is a removable singularity of f.*

Proof. By replacing the variable z by $\tilde{z} = z - a$, we can assume that $a = 0$ for simplicity. Define $h : D_r(0) \to \mathbb{C}$ as $h(z) := z^2 f(z)$ for $z \neq 0$ and $h(0) := 0$. Then h is complex differentiable on $D_r^\bullet(0)$, and

$$h'(0) = \lim_{z \to 0} \frac{h(z) - h(0)}{z - 0} = \lim_{z \to 0} \frac{z^2 f(z)}{z} = \lim_{z \to 0} z \cdot f(z) = 0$$

where we use that f is bounded in the last equality. This shows that $h : D_r(0) \to \mathbb{C}$ is holomorphic and thus equal to a power series $h(z) = \sum a_n z^n$ on $D_r(0)$. Since $h(0) = 0$ and $h'(0) = 0$, we have $a_0 = 0$ and $a_1 = 0$. So f is equal to the power series

$$f(z) \;=\; \frac{1}{z^2} \cdot h(z) \;=\; a_2 + a_3 z + \cdots$$

on $D_r^\bullet(0)$, which has order $\mathrm{ord}_0(f) \geqslant 0$. Therefore $a = 0$ is a removable singularity. □

Proposition 5.1.5. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic with pole a. Then*

$$\lim_{z \to a} |f(z)| \;=\; \infty.$$

Proof. If a is a pole of order $|m|$ (where $m = \mathrm{ord}_a(f) < 0$), then $f(z) = (z-a)^m h(z)$ for a holomorphic function $h : D_r(a) \to \mathbb{C}$ with $h(a) \neq 0$ for a sufficiently small punctured disc $D_r^\bullet(a) \subset U$. Thus

$$\lim_{z \to a} |f(z)| \;=\; |h(a)| \cdot \lim_{z \to a} |z-a|^m \;=\; |h(a)| \cdot \lim_{z \to a} \left(\frac{1}{|z-a|} \right)^{|m|} \;=\; \infty,$$

as claimed. □

Theorem 5.1.6 (Casaroti-Weierstrass)**.** *Let $f : D_r^\bullet(a) \to \mathbb{C}$ be holomorphic with essential singularity $a \in \mathbb{C}$. Then $f(D_r^\bullet(a))$ is dense in $\mathbb{C}$.*

Proof. We aim to lead the assumption $\overline{f(D_r^\bullet(a))} \neq \mathbb{C}$ to a contradiction. If there is a $b \in \mathbb{C} \setminus \overline{f(D_r^\bullet(a))}$, then there is an $s > 0$ such that $D_s(b) \cap f(D_r^\bullet(a)) = \varnothing$. Thus the function $g : D_r^\bullet(a) \to \mathbb{C}$ defined by

$$g(z) \;:=\; \frac{1}{f(z) - b}$$

is holomorphic and bounded by $\frac{1}{s}$. By Riemann's theorem on removable singularities (Theorem 4.1.4), g extends analytically to $D_r(a)$. In consequence,

$$f(z) \;=\; \frac{1}{g(z)} + b$$

has order

$$\mathrm{ord}_a(f) \;\geqslant\; \min\{\mathrm{ord}_a\left(\frac{1}{g}\right), \, \mathrm{ord}_a(b)\} = \min\{-\mathrm{ord}_a(g), 0\} \;>\; -\infty,$$

which is a contradiction to our assumption that a is essential. We conclude that $\overline{f(D_r^\bullet(a))} = \mathbb{C}$. □

In fact, the image $f(D_r^\bullet(a))$ of a punctured disc around an essential singularity a is characterized more accurately by the following strengthening of Theorem 4.1.6, which we state without proof.

Theorem 5.1.7 (Great Picard theorem)**.** *Let $f : D_r^\bullet(a) \to \mathbb{C}$ be holomorphic with essential singularity a. Then $f(D_r^\bullet(a))$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{b\}$ for some $b \in \mathbb{C}$.*

Conclusion. A singularity a of f is

$$\begin{aligned} \text{removable} \quad &\iff \quad f \text{ extends analytically to } a; \\ \text{a pole} \quad &\iff \quad \lim_{z \to a} |f(z)| \;=\; \infty; \\ \text{essential} \quad &\iff \quad f(D_r^\bullet(a)) \text{ is dense in } \mathbb{C}. \end{aligned}$$

5.2 Laurent expansions

Definition 5.2.1. A *Laurent series (about $c \in \mathbb{C}$)* is an expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-c)^n \quad = \quad \underbrace{\sum_{n=-\infty}^{-1} a_n(z-c)^n}_{\textit{(principal part)}} \quad + \quad \underbrace{\sum_{n=0}^{\infty} a_n(z-c)^n}_{\textit{(regular part)}}$$

with $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. The regular part $\sum_{n=0}^{\infty} a_n(z-c)^n$ of f is also called *analytic part* and *secondary part*. A Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-c)^n$ *converges (absolutely) at* $z \in \mathbb{C}$ if both principal and regular parts converge (absolutely) (where we consider the principal part as a series in $-n$).

Let $0 \leqslant s < r \leqslant \infty$ and $c \in \mathbb{C}$. The set

$$\mathrm{Ann}_{s,r}(c) \;:=\; \{z \in \mathbb{C} \mid s < |z-c| < r\}$$

is called an *(open) annulus (with center c)*.

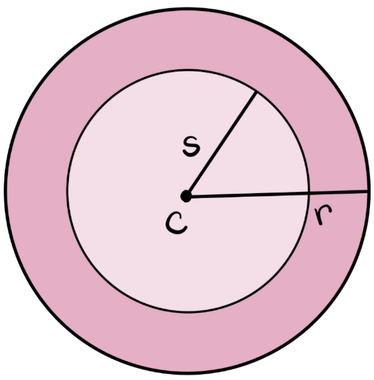


Figure 5.2: Annulus is the darker shaded area

Remark 5.2.2. (1) $D_r^\bullet(a) = \mathrm{Ann}_{0,r}(a)$.

(2) Let $\sum a_n(z-c)^n$ be a Laurent series whose principal part has radius of convergence $s^{-1} = \left(\limsup \sqrt[n]{|a_{-n}|}\right)^{-1}$ (for $n \geqslant 1$) and whose regular part has radius of convergence $r = \left(\limsup \sqrt[n]{|a_n|}\right)^{-1}$. Then

- the principal part $\sum_{n=-\infty}^{-1} a_n(z-c)^n$ converges absolutely for $|z-c|^{-1} < s^{-1}$;
- the regular part $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges absolutely for $|z-c| < r$.

Thus if $s < r$, then the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-c)^n$ converges absolutely for all $z \in \mathrm{Ann}_{s,r}(a)$.

Example 5.2.3. (1) The Laurent series z^{-1} has $s = 0$ (where $s^{-1} = \infty$ is the radius of convergence of the principal part z^{-1}) and radius of convergence $r = \infty$ for the regular part 0, thus z^{-1} converges absolutely on $\mathrm{Ann}_{0,\infty}(0) = \mathbb{C}^\times$.

(2) The Laurent series

$$e^{1/z} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \;=\; \sum_{n=-\infty}^{-1} \frac{1}{|n|!}\, z^n \;+\; 1$$

has $s = \frac{1}{\infty} = 0$ and $r = \infty$, and thus converges absolutely on $\mathrm{Ann}_{0,\infty}(0) = \mathbb{C}^\times$.

(3) The Laurent series

$$\sum_{n=-\infty}^{-1} z^n \;+\; \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

has $s = 1$ and $r = 2$, and thus converges on $\mathrm{Ann}_{1,2}(0)$.

Theorem 5.2.4 (Laurent expansion on annuli)**.** *Let $0 \leqslant s < t < r \leqslant \infty$ and $c \in \mathbb{C}$. Let $f : \mathrm{Ann}_{s,r}(c) \to \mathbb{C}$ be holomorphic and define*

$$a_n \;:=\; \frac{1}{2\pi\mathbf{i}} \cdot \int_{C_t(c)} \frac{f(w)}{(w-c)^{n+1}}\, dw$$

for $n \in \mathbb{Z}$. Then

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n(z-c)^n$$

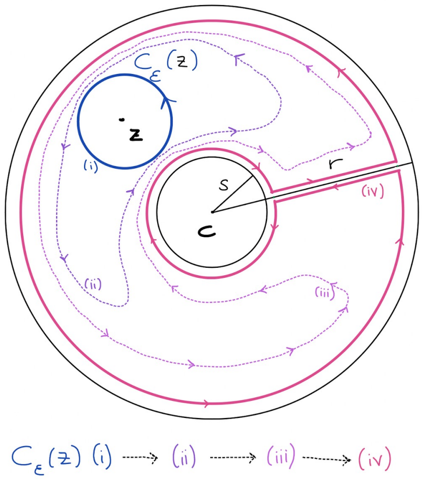
for all $z \in \mathrm{Ann}_{s,r}(c)$. In particular, the right hand side converges for all $z \in \mathrm{Ann}_{s,r}(c)$.

Remark 5.2.5. Before we turn to the proof of the theorem, let us observe that if f extends analytically to $D_r(c)$, then Cauchy's integral theorem (Theorem 2.7.8) implies that for $n \leqslant -1$,

$$a_n \;=\; \frac{1}{2\pi\mathbf{i}} \cdot \int_{C_t(c)} f(z) \,\cdot\, (z-c)^{|n|-1}\, dz \;=\; 0.$$

Thus $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ agrees with the Taylor expansion in this case.

Proof. Consider $z \in \mathrm{Ann}_{s,r}(c)$ and $\gamma_0 = C_\epsilon(z)$ with $\mathrm{im}\,\gamma_0 \subset \mathrm{Ann}_{s,r}(c)$.



Then γ_0 is closed homotopic in $\mathrm{Ann}_{s,r}(c)$ to $\gamma_1 = C_{r'}(c) + \tilde{\gamma} + C_{s'}(c)^- + \tilde{\gamma}^-$ where $|z-c| + \epsilon < r' < r$ and $s < s' < |z-c| - \epsilon$ and $\tilde{\gamma}$ is a linear path from $c + r'$ to $c + s'$, as illustrated above. Thus

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\mathbf{i}} \int\limits_{\gamma_0} \frac{f(w)}{w-z}\, dw \\
&= \frac{1}{2\pi\mathbf{i}} \int\limits_{\gamma_1} \frac{f(w)}{w-z}\, dw \\
&= \frac{1}{2\pi\mathbf{i}} \left(\int\limits_{C_{r'}(c)} \frac{f(w)}{w-z}\, dw + \int\limits_{\tilde{\gamma}} \frac{f(w)}{w-z}\, dw - \int\limits_{C_{s'}(c)} \frac{f(w)}{w-z}\, dw - \int\limits_{\tilde{\gamma}} \frac{f(w)}{w-z}\, dw \right) \\
&= \frac{1}{2\pi\mathbf{i}} \left(\int\limits_{C_{r'}(c)} \frac{f(w)}{w-z}\, dw - \int\limits_{C_{s'}(c)} \frac{f(w)}{w-z}\, dw \right)
\end{aligned}$$

where the first equality follows from Cauchy's integral formula (Theorem 3.1.1) and the second equality follows from Cauchy's integral theorem (Theorem 2.7.4). As in the proof of Theorem 3.4.3, we compute

$$\begin{aligned}
\int\limits_{C_{r'}(c)} \frac{f(w)}{w-z}\, dw &= \int\limits_{C_{r'}(c)} \frac{f(w)}{w-c} \cdot \frac{1}{1 - \frac{z-c}{w-c}}\, dw \\
&= \int\limits_{C_{r'}(c)} \frac{f(w)}{w-c} \cdot \sum_{n=0}^{\infty} \left(\frac{z-c}{w-c} \right)^n dw \\
&= \sum_{n=0}^{\infty} \left(\int\limits_{C_t(c)} \frac{f(w)}{(w-c)^{n+1}}\, dw \right) \cdot (z-c)^n \\
&= 2\pi\mathbf{i} \sum_{n=0}^{\infty} a_n (z-c)^n
\end{aligned}$$

where $\frac{z-c}{w-c} \in D_1(0)$ for $w \in \operatorname{im} C_{r'}(c)$ and therefore $\sum_{n=0}^{\infty} \left(\frac{z-c}{w-c}\right)^n$ converges absolutely, which allows us to exchange the integral and summation by Lemma 3.4.2. Since $\frac{f(w)}{(w-c)^{n+1}}$ is defined for all $w \in \mathrm{Ann}_{s,r}(c)$, we can use Cauchy's integral theorem (for closed homotopies, Theorem 2.7.4) to exchange $C_{r'}(c)$ by $C_t(c)$ without changing the value of the path integral in the third equality.

Similarly we have

$$\begin{aligned} -\int_{C_{s'}(c)} \frac{f(w)}{w-z}\,dw &= \int_{C_{s'}(c)} \frac{f(w)}{z-c} \cdot \frac{1}{1-\frac{w-c}{z-c}}\,dw \\ &= \int_{C_{s'}(c)} \frac{f(w)}{z-c} \cdot \sum_{n=0}^{\infty} \left(\frac{w-c}{z-c}\right)^n dw \\ &= \sum_{n=0}^{\infty} \left(\int_{C_t(c)} \frac{f(w)}{(w-c)^{-n}}\,dw \right) \cdot (z-c)^{-(n+1)} \\ &= \sum_{n=-\infty}^{-1} \left(\int_{C_t(c)} \frac{f(w)}{(w-c)^{n+1}}\,dw \right) \cdot (z-c)^n \\ &= 2\pi\mathbf{i} \sum_{n=-\infty}^{-1} a_n (z-c)^n \end{aligned}$$

since $\frac{w-c}{z-c} \in D_1(0)$ for $w \in \operatorname{im} C_{s'}(c)$, and where we use exchange n by $-(n+1)$ in the fourth equality. Putting the regular and principal part together yields

$$f(z) = \frac{1}{2\pi\mathbf{i}} \left(\int_{C_{r'}(c)} \frac{f(w)}{w-z}\,dw - \int_{C_{s'}(c)} \frac{f(w)}{w-z}\,dw \right) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n,$$

as claimed. In particular, this shows that $\sum a_n (z-c)^n$ converges for all $z \in \mathrm{Ann}_{s,r}(c)$. □

Definition 5.2.6. Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic, $c \in \mathbb{C}$, $0 < t < r$ and $D_r^\bullet(c) \subset U$. Then $f(z) = \sum_{n\in\mathbb{Z}} a_n (z-c)^n$ with

$$a_n := \frac{1}{2\pi\mathbf{i}} \int_{C_t(c)} \frac{f(w)}{(w-c)^{n+1}}\,dw$$

is called the *Laurent expansion of f at c*.

Remark 5.2.7. (1) The equality $f(z) = \sum a_n (z-c)^n$ for $z \in \mathrm{Ann}_{s,r}(c)$ (as in Theorem 4.2.4) determines the coefficients a_n uniquely; cf. Exercise 4.8.5.

(2) The Laurent expansions of Theorem 4.2.4 of the same function can differ for annuli $\mathrm{Ann}_{s,r}(c)$ and $\mathrm{Ann}_{s',r'}(c)$ with empty intersection $\mathrm{Ann}_{s,r}(c) \cap \mathrm{Ann}_{s',r'}(c) = \varnothing$. For example,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

for $z \in D_1^\bullet(0) = \mathrm{Ann}_{0,1}(0)$, and

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{-1} (-1) \cdot z^n$$

for $z \in \mathrm{Ann}_{1,\infty}(0)$.

Proposition 5.2.8. *Let $c \in \mathbb{C}$, $r > 0$ and $f : D_r^\bullet(c) \to \mathbb{C}$ be a nontrivial holomorphic function with Laurent expansion $f(z) = \sum a_n(z-c)^n$ at c. Then*

$$\mathrm{ord}_c(f) = \inf\{n \in \mathbb{Z} \mid a_n \neq 0\}.$$

Proof. If $m = \mathrm{ord}_c(f) \neq -\infty$, then

$$f(z) = (z-c)^m \sum_{n=0}^{\infty} \tilde{a}_n(z-c)^n = \sum_{n=m}^{\infty} a_n(z-c)^n$$

for $z \in D_r^\bullet(c)$ and $\tilde{a}_{n-m} = a_n$ where $a_m = \tilde{a}_0 \neq 0$. Thus

$$\mathrm{ord}_c(f) = m = \inf\{n \in \mathbb{Z} \mid a_n \neq 0\},$$

as claimed.

If $\mathrm{ord}_c(f) = -\infty$, then there is no $m \in \mathbb{Z}$ such that

$$h(z) = (z-c)^{-m} \sum_{n=-\infty}^{\infty} a_n(z-c)^n$$

extends analytically to c. Thus

$$\mathrm{ord}_c(f) = -\infty = \inf\{n \in \mathbb{Z} \mid a_n \neq 0\}.$$ □

5.3 Residues

Definition 5.3.1. Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic, $D_r^\bullet(c) \subset U$ (for some $r > 0$ and $c \in \mathbb{C}$) and $f(z) = \sum a_n(z-c)^n$ the Laurent expansion of f at c. The *residue of f at c* is

$$\mathrm{Res}_c f = a_{-1}.$$

Remark 5.3.2. The relevance of the residue becomes apparent in the formula

$$\int_{C_t(c)} f = 2\pi\mathbf{i} \cdot \mathrm{Res}_c f$$

for $0 < t < r$, which follows at once from the definition of the residue and Theorem 4.2.4.

We study some methods to compute the residue: First of all note that if $c \in U$, then $\sum a_n(z-c)^n$ is a power series and thus $\mathrm{Res}_c f = 0$. If c is a pole, then we can compute the residue in terms of derivatives as follows.

Lemma 5.3.3. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic and $D_r^\bullet(c) \subset U$. Assume that c is a pole of f of order $m \geqslant 1$. Then*

$$\mathrm{Res}_c f \;=\; \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \Big[(z-c)^m\, f(z)\Big]_{z=c}.$$

Proof. By Proposition 4.2.8, $-m$ is the index of the smallest non-vanishing coefficient $a_{-m} \neq 0$ of the Laurent expansion $f(z) = \sum a_n (z-c)^n$ of f at c. Thus

$$(z-c)^m\, f(z) \;=\; a_{-m} \;+\; a_{-m+1}(z-c) \;+\; \cdots \;+\; a_{-1}(z-c)^{m-1} \;+\; \cdots$$

has derivative

$$\frac{d^{m-1}}{dz^{m-1}} \Big[(z-c)^m\, f(z)\Big]_{z=c} \;=\; (m-1)! \cdot a_{-1} \;=\; (m-1)! \cdot \mathrm{Res}_c f$$

at $z = c$. □

Remark 5.3.4. In fact, the same reasoning applies to any other coefficient of the Laurent expansion of f at c, under the assumption that $-m = \mathrm{ord}_c(f) \neq -\infty$ (cf. Exercise 4.8.7). Then we have

$$a_k \;=\; \frac{1}{(m+k)!} \cdot \frac{d^{m+k}}{dz^{m+k}} \Big[(z-c)^m\, f(z)\Big]_{z=c}$$

for $k \geqslant -m$.

Example 5.3.5. The rational function $f : \mathbb{C} \setminus \{0,1\} \to \mathbb{C}$ with

$$f(z) \;=\; \frac{1}{z^2(z-1)}$$

has $\mathrm{ord}_0(f) = -2$ and $\mathrm{ord}_1(f) = -1$. By Lemma 4.3.3, we have

$$\begin{aligned} \mathrm{Res}_0 f \;&=\; \frac{1}{1!} \cdot \frac{d}{dz} \Big[(z-0)^2 \frac{1}{z^2(z-1)}\Big]_{z=0} \\ &=\; \frac{d}{dz}\Big[\frac{1}{z-1}\Big]_{z=0} \;=\; \Big[\frac{-1}{(z-1)^2}\Big]_{z=0} \;=\; -1 \end{aligned}$$

and

$$\mathrm{Res}_1 f \;=\; \frac{1}{0!} \cdot \frac{d^0}{dz^0} \Big[(z-1)^1 \frac{1}{z^2(z-1)}\Big]_{z=1} \;=\; \Big[\frac{1}{z^2}\Big]_{z=1} \;=\; 1.$$

Thus by Remark 4.3.2, we have

$$\int\limits_{C_{1/2}(0)} f \;=\; -2\pi\mathbf{i} \qquad \text{and} \qquad \int\limits_{C_{1/2}(1)} f \;=\; 2\pi\mathbf{i}.$$

5.4 The residue theorem

Definition 5.4.1. Let $\gamma : I \to \mathbb{C}$ be a closed path and $c \in \mathbb{C} \setminus \operatorname{im}\gamma$. The *winding number of* γ *around* c is

$$W(\gamma, c) \;=\; \frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{1}{w-c}\, dw.$$

Example 5.4.2. Let $\gamma_i : I \to \mathbb{C}$ (for $i = 1, 2$) be given by $\gamma_1(t) = e^{2\pi\mathbf{i}\cdot t}$ and $\gamma_2(t) = e^{-4\pi\mathbf{i}\cdot t}$:

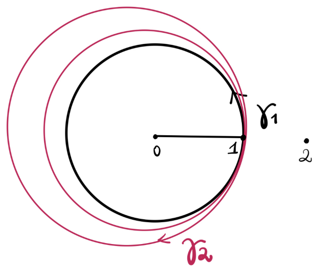


Figure 5.3: Since γ_2 moves in the opposite direction than γ_1 the winding number around 0 is negative

Then

$$W(\gamma_1, 0) \;=\; 1, \qquad W(\gamma_1, 2) \;=\; 0 \qquad \text{and} \qquad W(\gamma_2, 0) \;=\; -2.$$

Remark 5.4.3. The winding number satisfies the following properties:

(0) If γ is a constant path, then $W(\gamma, c) = 0$ for any $c \in \mathbb{C} \setminus \operatorname{im}\gamma$.

(1) The winding number is an integer $W(\gamma, c) \in \mathbb{Z}$ for any $\gamma : I \to \mathbb{C}$ and $c \in \mathbb{C} \setminus \operatorname{im}\gamma$.

(2) If γ^- is the inverse path to γ (defined by $\gamma^-(t) = \gamma(1-t)$). Then

$$W(\gamma^-, c) \;=\; -W(\gamma, c).$$

(3) Let $\gamma = \gamma_1 + \cdots + \gamma_n$ be the concatenation of n closed paths $\gamma_1, \ldots, \gamma_n$ and assume that $c \notin \operatorname{im}\gamma_i$ for all $i = 1, \ldots, n$. Then

$$W(\gamma, c) \;=\; \sum_{i=1}^{n} W(\gamma_i, c).$$

(4) If $\gamma_0 \sim \gamma_1$ are closed homotopic in $\mathbb{C} \setminus \{c\}$, then

$$W(\gamma_0, c) \;=\; W(\gamma_1, c).$$

Theorem 5.4.4 (Residue theorem). *Let $V \subset \mathbb{C}$ be open and $U = V \setminus \{c_1, \ldots, c_s\}$ for pairwise distinct points $c_1, \ldots, c_s \in V$. Let $\gamma : I \to U$ be a closed path that is contractible in V. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then*

$$\int_\gamma f \;=\; 2\pi\mathbf{i} \cdot \sum_{i=1}^{s} W(\gamma, c_i) \cdot \operatorname{Res}_{c_i} f.$$

Proof. We restrict ourselves to the case that none of the c_i are an essential singularity of f since the proof for essential singularities is more complicated.

For $i = 1, \dots, s$, let

$$f(z) \;=\; \sum_{n=m_i}^{\infty} a_{i,n}(z-c_i)^n$$

be the Laurent expansion of f at c_i where $m_i = \mathrm{ord}_{c_i}(f) > -\infty$. Note that its principal part $\sum_{n=m_i}^{-1} a_{i,n}(z-c_i)^n$ defines a holomorphic function on $\mathbb{C} \setminus \{c_i\}$. Since the Laurent expansion of

$$g(z) \;=\; f(z) \;-\; \sum_{i=1}^{s} \sum_{n=m_i}^{-1} a_{i,n}(z-c_i)^n$$

at each of $c_1, \dots, c_s$ is a power series by construction, the holomorphic function $g : U \to \mathbb{C}$ extends analytically to V. Since γ is contractible in V, we have $\int_\gamma g = 0$ by Cauchy's integral theorem (Theorem 2.7.6). Thus

$$\begin{aligned} \int_\gamma f \;&=\; \sum_{i=1}^{s} \sum_{n=m_i}^{-1} a_{i,n} \int_\gamma (w-c_i)^n \, dw \\ &=\; \sum_{i=1}^{s} a_{i,-1} \int_\gamma (w-c_i)^{-1} \, dw \\ &=\; 2\pi\mathbf{i} \cdot \sum_{i=1}^{s} W(\gamma, c_i) \cdot \mathrm{Res}_{c_i} f. \end{aligned}$$

where we use that $\int_\gamma (w-c_i)^n \, dw = 0$ for $n \neq -1$ in the second equality, and insert the definitions of the winding number and the residue in the last equality. □

Example 5.4.5. We continue Example 4.3.5: consider the rational function given by

$$f(z) \;=\; \frac{1}{z^2(z-1)}$$

and the path $\gamma : I \to \mathbb{C}$ given by

$$\gamma(t) \;=\; \begin{cases} \frac{1}{2} + e^{4\pi\mathbf{i}\cdot t} & \text{for } 0 \leqslant t \leqslant \frac{1}{2}; \\ 1 + \frac{1}{2} \cdot e^{4\pi\mathbf{i}\cdot t} & \text{for } \frac{1}{2} < t \leqslant 1. \end{cases}$$

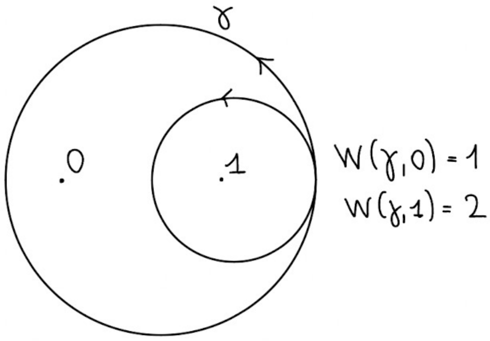


Figure 5.4: The winding number around zero is 1 since we only go around it once, but around 1 we circle two times, hence the winding number is 2.

Then, with the notation as in Theorem 4.4.4, we let $V = \mathbb{C}$ and $U = \mathbb{C} \setminus \{0,1\}$ to get

$$\int_\gamma f \;=\; 2\pi\mathbf{i}\,\Big(W(\gamma,0)\,\cdot \mathrm{Res}_0 f \;+\; W(\gamma,1)\,\cdot\,\mathrm{Res}_1 f\Big)$$
$$= 2\pi\mathbf{i}\,\cdot\,(1\cdot(-1)\,+\,2\cdot 1)\;=\;2\pi\mathbf{i}.$$

5.5 The argument principle

Definition 5.5.1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic. Let $S := f^{-1}(0) \subset U$ be the set of zeros of f. The *logarithmic derivative of* f is the holomorphic function

$$\frac{f'}{f} : U \setminus S \longrightarrow \mathbb{C}.$$

Remark 5.5.2.

(1) Let $\log_\alpha$ be any branch of the logarithm and assume that $U \setminus S \subset U_\vartheta$ for $\vartheta = e^{\mathbf{i}\alpha}$. Then

$$\frac{f'(z)}{f(z)} \;=\; \frac{d}{dz}\,\log_\alpha f(z)$$

for all $z \in U \setminus S$.

(2) Assume that $g : U \setminus S \to \mathbb{C}$ is holomorphic and without zeros. Then

$$\frac{(fg)'}{fg} \;=\; \frac{f'g + fg'}{fg} \;=\; \frac{f'}{f} \;+\; \frac{g'}{g}$$

as functions on $U \setminus S$, i.e. the logarithmic derivative of a product of functions is the sum of the logarithmic derivatives of the factors.

Proposition 5.5.3. *Let $f : D_r^\bullet(c) \to \mathbb{C}$ be holomorphic. Assume that* $\operatorname{ord}_c(f) \neq -\infty$ *and that $f(z) \neq 0$ for all $z \in D_r^\bullet(c)$. Then there is a holomorphic function $g : D_r(c) \to \mathbb{C}$ such that*

$$\frac{f'}{f}(z) \;=\; \frac{\operatorname{ord}_c(f)}{z-c} \;+\; g(z)$$

for all $z \in D_r^\bullet(c)$. In particular, $\operatorname{ord}_c(f) = \operatorname{Res}_c(f'/f)$.

Proof. Let and $f(z) = \sum a_n(z-c)^n$ the Laurent expansion of f at c and $m = \operatorname{ord}_c(f)$. By Proposition 4.2.8,

$$f'(z) \;=\; m \cdot a_m (z-c)^{m-1} \;+\; \cdots$$

has $\operatorname{ord}_c(f') \geqslant m-1$, and thus by Remark 4.1.2 we have

$$\operatorname{ord}_c(f'/f) \;=\; \operatorname{ord}_c(f') \;-\; \operatorname{ord}_c(f) \;\geqslant\; m-1-m \;=\; -1.$$

By the definition of $\operatorname{ord}_c(f)$, there exists a holomorphic function $h : D_r(c) \to \mathbb{C}$ with

$$f(z) \;=\; (z-c)^m \;\cdot\; h(z)$$

for all $z \in D_r^\bullet(c)$. Since neither $f(z)$ nor $(z-c)^m$ has a zero in $D_r^\bullet(c)$, also h does not have a zero in $D_r^\bullet(c)$. Thus $g = \frac{h'}{h}$ is holomorphic on $D_r^\bullet(c)$, and

$$\frac{f'}{f} \;=\; \frac{\big((z-c)^m h\big)'}{(z-c)^m h} \;=\; \frac{m(z-c)^{m-1}}{(z-c)^m} \;+\; \frac{h'}{h} \;=\; \frac{\operatorname{ord}_c(f)}{z-c} \;+\; g$$

where we use Remark 4.5.2 in the second equality and $m = \operatorname{ord}_c(f)$ in the third equality. This concludes the proof of the first claim. The second claim follows from the fact that the Laurent expansion of $\frac{f'}{f}$ at c is of the form

$$\operatorname{ord}_c(f) \cdot (z-c)^{-1} \;+\; \sum_{n=0}^{\infty} b_n (z-c)^n$$

where $g(z) = \sum_{n=0}^{\infty} b_n (z-c)^n$ is the Taylor expansion of g at c. □

Theorem 5.5.4 (The argument principle)**.** *Let $U \subset V \subset \mathbb{C}$ be open subsets such that $S = V \setminus U$ is finite. Let $f : U \to \mathbb{C}$ be a nontrivial holomorphic function with* $\operatorname{ord}_c(f) \neq -\infty$ *for all $c \in S$. Let $\gamma : I \to U$ be a closed path that is contractible in V and such that $f(\gamma(t)) \neq 0$ for all $t \in I$. Then*

$$\sum_{c \in S \cup \{\text{zeros of } f\}} W(\gamma, c) \;\cdot\; \operatorname{ord}_c(f) \quad=\quad \frac{1}{2\pi \mathbf{i}} \cdot \int_\gamma \frac{f'}{f} \quad=\quad W(f \circ \gamma, 0).$$

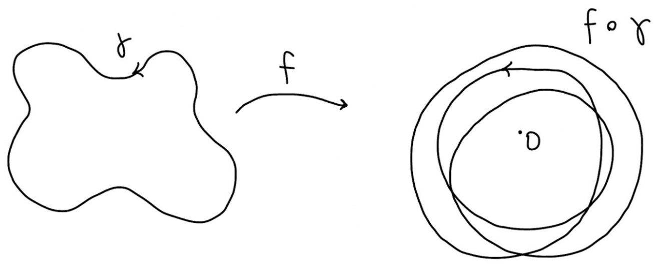


Figure 5.5: Closed path maps to closed path

Proof. This follows from the direct computation

$$\begin{aligned}\sum_{c\in S\cup\{\text{zeros of } f\}} W(\gamma,c)\cdot \operatorname{ord}_c(f) &= \sum_{c\in S\cup\{\text{zeros of } f\}} W(\gamma,c)\cdot \operatorname{Res}_c \frac{f'}{f}\\ &= \frac{1}{2\pi\mathbf{i}}\cdot\int_\gamma \frac{f'}{f}\\ &= \frac{1}{2\pi\mathbf{i}}\cdot\int_\gamma \frac{1}{f}\cdot\frac{df}{dz}\,dz\\ &= \frac{1}{2\pi\mathbf{i}}\cdot\int_{f\circ\gamma}\frac{1}{w}\,dw\\ &= W(f\circ\gamma,0),\end{aligned}$$

whose first equality holds by Proposition 4.5.3, whose second equality holds by the residue theorem (Theorem 4.4.4) and whose fourth equality follows from a variable substitution of z by $w=f(z)$. □

Example 5.5.5. Consider $\gamma = C_r(0)$, $V=\mathbb{C}$, $U=\mathbb{C}^\times$, and $f(z)=z^n$. Then $\operatorname{ord}_0(f)=n$ and the closed path $f\circ C_r(0): I\to\mathbb{C}$ (given by $t\mapsto r^n\cdot e^{2\pi\mathbf{i}\cdot nt}$) circles n-times counter-clockwise around 0. Thus we find back the result of the argument principle (Theorem 4.5.4):

$$W(f\circ C_r(0),0) \;=\; n \;=\; 1\cdot n \;=\; W(C_r(0),0)\cdot\operatorname{ord}_0(f).$$

We can also determine the third quantity explicitly as

$$\frac{1}{2\pi\mathbf{i}}\int_{C_r(0)}\frac{f'}{f} \;=\; \frac{1}{2\pi\mathbf{i}}\int_{C_r(0)}\frac{n}{z}\,dz \;=\; \frac{2\pi\mathbf{i}\cdot n}{2\pi\mathbf{i}} \;=\; n$$

in this case.

Definition 5.5.6. Let $\gamma: I\to\mathbb{C}$ be a closed path and $U=\mathbb{C}\setminus\operatorname{im}\gamma$. The *interior of* γ is the union γ^{int} of all bounded connected components of U.

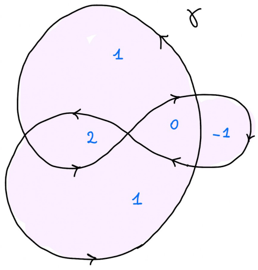


Figure 5.6: Shaded in pink is the union of γ interior components

Remark 5.5.7.

(1) Since $\operatorname{im}\gamma$ is bounded, γ^{int} is bounded and $U = \mathbb{C} \setminus \operatorname{im}\gamma$ has precisely one connected component that is unbounded.

(2) If $W(\gamma, z) \neq 0$ for $z \in U$, then $z \in \gamma^{\text{int}}$.

(3) It follows from Theorem 5.1.3 (proven later) that the set S_0 of zeros of a holomorphic function f is discrete. So $S_0 \cap \gamma^{\text{int}}$ is finite. Similarly, the set of poles of f inside γ^{int} is finite.

Corollary 5.5.8. *Let $U \subset V \subset \mathbb{C}$ be open subsets, $f : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ be a closed path that is contractible in V. Assume that $f(\gamma(t)) \neq 0$ for all $t \in I$, that $W(\gamma, z) = 1$ for all $z \in \gamma^{\text{int}}$, that $V \setminus U$ is finite, and that $\operatorname{ord}_c(f) \neq -\infty$ for all $c \in V \setminus U$. Define*

$$N(0) \;=\; \sum_{\textit{zeros } c \in \gamma^{\text{int}}} \operatorname{ord}_c(f) \qquad \textit{and} \qquad N(\infty) \;=\; \sum_{\textit{poles } c \in \gamma^{\text{int}}} |\operatorname{ord}_c(f)|.$$

Then

$$N(0) \,-\, N(\infty) \;=\; W(f \circ \gamma, 0).$$

Proof. This follows at once from the definitions of $N(0)$ and $N(\infty)$, the assumption that $W(\gamma, z) = 1$ for $z \in \gamma^{\text{int}}$, and the argument principle (Theorem 4.5.4):

$$N(0) \,-\, N(\infty) \;=\; \sum_{\substack{c \in \gamma^{\text{int}} \text{ with} \\ \operatorname{ord}_c(f) \neq 0}} W(\gamma, z) \cdot \operatorname{ord}_c(f) \;=\; W(f \circ \gamma, 0). \qquad \square$$

Remark 5.5.9. Note that the set S in Corollary 4.5.8 is the set of zeros and isolated singularities of f in the interior of γ. Since we exclude essential singularities, S contains only zeros, poles, and possibly removable singularities. The quantities $N(0)$ and $N(\infty)$ are the number of zeros and poles of f, respectively, counted with multiplicities.

Note that by Theorem 5.1.3 (as proven later) that the set of zeros of a holomorphic function $f : U \to \mathbb{C}$ is discrete. Since γ^{int} is bounded (cf. Remark 4.5.7), f has only finitely many zeros in γ^{int}. So the hypothesis that S (as defined in the theorem) is finite is automatic if we assume that $V \setminus U$ is finite.

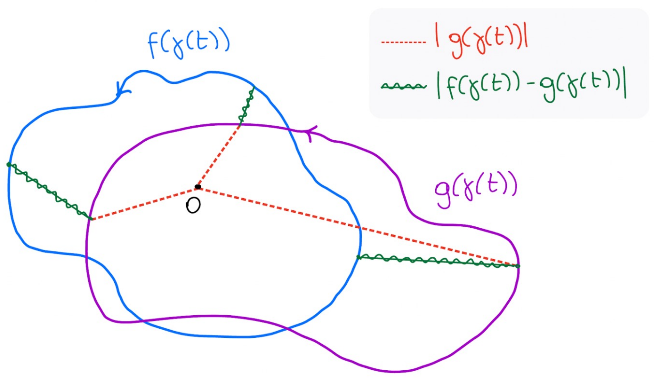
5.6 Rouché's theorem

Theorem 5.6.1 (Rouché's theorem)**.** *Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ a closed path that is contractible in U and such that $W(\gamma, z) = 1$ for all $z \in \gamma^{\text{int}}$. If*

$$\big|f(\gamma(t)) - g(\gamma(t))\big| \;<\; \big|g(\gamma(t))\big|$$

for all $t \in I$, then

$$\sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(f) \quad = \quad \sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(g).$$



Remark 5.6.2. As explained in Remark 4.5.9, the number of zeros and poles of f and g in γ^{int} is finite, so the sums in Rouché's theorem are finite (and thus well-defined) if we ignore the zero terms.

In fact, since γ is contractible in U, f does not have a singularity in γ^{int}, and thus $\operatorname{ord}_c(f) \geqslant 0$ for all $c \in \gamma^{\text{int}}$. This means that Rouché's theorem expresses an equality between the zeros of f and the zeros of g, when counted with multiplicities.

Proof. We define the continuous function

$$\begin{array}{rccl} H: & [0,1] \times U & \longrightarrow & \mathbb{C} \\ & (t,z) & \longmapsto & H_t(z) := g(z) + t(f(z) - g(z)) \end{array}$$

(a *homotopy from $g = H_0$ to $f = H_1$*). By the triangle inequality, we have

$$|a+b| \;+\; |b| \;=\; |a+b| \;+\; |-b| \;\geqslant\; |a+b-b| \;=\; |a|,$$

and thus $|a+b| \geqslant |a| - |b|$. Applying this to $a = g(z)$ and $b = t(f(z) - g(z))$ yields for all $z \in \operatorname{im}\gamma$ and $t \in I$ that

$$\begin{aligned} |H_t(z)| \;=\; \big|g(z) + t(f(z) - g(z))\big| \;&\geqslant\; |g(z)| \;-\; |t| \cdot |f(z) - g(z)| \\ &\geqslant\; |g(z)| \;-\; |f(z) - g(z)| \;>\; 0, \end{aligned}$$

where the second inequality follows from $|t| \leqslant 1$ and the last inequality follows from the hypothesis that $|f(z) - g(z)| < |g(z)|$ for all $z \in \operatorname{im}\gamma$.

This shows that $H_t(z) \neq 0$ for all $t \in I$ and $z \in \operatorname{im} \gamma$. Therefore Theorem 4.5.4 applies and yields (for any fixed $t \in I$)

$$N(t) \;:=\; \sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(H_t) \;=\; W(H_t \circ \gamma, 0) \;=\; \frac{1}{2\pi \mathbf{i}} \cdot \int_{H_t \circ \gamma} \frac{1}{w}\, dw$$

which is an integer that is independent of t by Cauchy's integral theorem (Theorem 2.7.4) since $H_t \circ \gamma : I \to \mathbb{C}^\times$ is a closed homotopy in $\mathbb{C}^\times$, the domain of $\frac{1}{w}$. Thus

$$\sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(f) \;=\; N(1) \;=\; N(0) \;=\; \sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(g),$$

as claimed. □

As an application, we find another short proof of the fundamental theorem of algebra (Theorem 3.6.1).

Theorem 5.6.3. *Every complex polynomial of positive degree has a zero.*

Proof. Consider a complex polynomial $f = a_d z^d + \cdots + a_1 z + a_0$ of degree $d \geqslant 1$. Define $g = a_d z^d$ and let $r > 0$ be sufficiently large, so that

$$|f(z) - g(z)| \;=\; |a_{d-1} z^{d-1} + \cdots + a_0| \;<\; |a_d z^d| \;=\; |g(z)|$$

for all $z \in \operatorname{im} C_r(0)$. Then

$$\sum_{c \in C_r(0)^{\text{int}}} \operatorname{ord}_c(f) \;=\; \sum_{c \in C_r(0)^{\text{int}}} \operatorname{ord}_c(g) \;=\; \operatorname{ord}_0(g) \;=\; d$$

by Rouché's theorem (Theorem 4.6.1), which shows that f has $d > 0$ zeros, counted with multiplicities. □

Example 5.6.4. In fact, the method of the preceding proof provides a constructive method to narrow down the location of the zeros of a complex polynomial. We demonstrate this in the following example of the polynomial $f = z^5 + 4z + 2$.

(1) Let $g = z^5$. Then for $|z| = 2$, we have

$$|f(z) - g(z)| \;=\; |4z + 2| \;\leqslant\; 4 \cdot |z| + 2 \;=\; 10 \;<\; 32 \;=\; |z|^5 \;=\; |g(z)|,$$

which allows us to apply Rouché's theorem to conclude that

$$\sum_{|c|<2} \operatorname{ord}_c(f) \;=\; \operatorname{ord}_0(g) \;=\; 5.$$

Thus all 5 zeros of f (counted with multiplicities) lie inside the open disc $D_2(0)$.

(2) Let $\tilde{g}(z) = 4z$. Then for $|z| = 1$, we have

$$|f(z) - \tilde{g}(z)| \ = \ |z^5 + 2| \ \leqslant \ |z|^5 + 2 \ < \ 4 \ = \ 4 \cdot |z| \ = \ |\tilde{g}(z)|.$$

Thus by Rouché's theorem, we have

$$\sum_{|c|<1} \mathrm{ord}_c(f) \ = \ \mathrm{ord}_0(\tilde{g}) \ = \ 1,$$

which shows that f has precisely one simple zero (multiplicity 1) in the open disc $D_1(0)$.

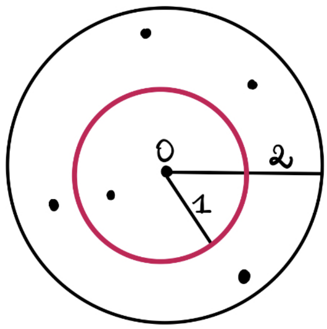


Figure 5.7: Open disc with $r = 1$ with one zero inside and open disc with $r = 2$ with five zeros inside

5.7 The open mapping principle

Let $U \subset \mathbb{C}$ be open. A holomorphic function $f : U \to \mathbb{C}$ is *locally non-constant* if for every open disc $D_r(a)$ that is contained in U, the restriction of f to $D_r(a)$ is non-constant.

Theorem 5.7.1 (open mapping principle)**.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic and locally non-constant. Then $f(U)$ is open.*

Proof. The image $f(U)$ is open if for every $a \in U$ and $b = f(a)$, there is an $\epsilon > 0$ such that $D_\epsilon(b) \subset f(U)$.

To find such an ϵ, we define for fixed $a \in U$ and $b = f(a)$ the function $f_b(z) = f(z) - b$. Since $f_b(a) = 0$, we have that $m = \mathrm{ord}_a(f_b) \geqslant 1$. Let $r > 0$ be such that $D_r(a)$ is contained in U. By the definition of $\mathrm{ord}_a(f_b)$, there is a holomorphic function $g : D_r(a) \to \mathbb{C}$ with $g(a) \neq 0$ and

$$f_b(z) \ = \ (z-a)^m \cdot g(z)$$

for all $z \in D_r(a)$. Since g is continuous, there is an $s \in (0, r)$ such that $g(z) \neq 0$ for all $z \in \overline{D}_s(a)$. Since $\partial\overline{D}_s(a) = \{z \in \mathbb{C} \mid |z-a| = s\}$ is compact, $|f_b|$ assumes its minimum on $\partial\overline{D}_s(a)$. Now set

$$\epsilon \ := \ \min\big\{\, |f_b(z)| \,\big|\, |z-a| = s \,\big\} \ > \ 0.$$

Note that $\epsilon > 0$ is guaranteed since $(z-a)^m \neq 0$ and $g(z) \neq 0$ for all $z \in \partial\overline{D}_s(a)$. We claim that $D_\epsilon(b) \subset f(U)$, which implies that $f(U)$ is open.

Consider $w \in D_\epsilon(b)$. For $|z-a| = s$, we have

$$\big|(f(z)-w)-f_b(z)\big| \;=\; |w-b| \;<\; \epsilon \;\leqslant\; |f_b(z)|.$$

It then follows by Rouché's theorem (Theorem 4.6.1) that

$$\sum_{|c-a|<s} \mathrm{ord}_c(f(z)-w) \;=\; \sum_{|c-a|<s} \mathrm{ord}_c(f_b) \;>\; 0.$$

Since $f_b(a) = 0$, we hence have $f(z)-w = 0$ for some $z \in D_s(a)$. Thus $w = f(z)$ is in $f(U)$, which shows that $D_\epsilon(b) \subset f(U)$ as claimed. □

As a consequence of the open mapping principle, we can remove the hypothesis that the inverse of a holomorphic bijection is continuous from the inverse function theorem (Theorem 1.2.5).

Theorem 5.7.2 (inverse function theorem)**.** *Let U and V be open subsets of $\mathbb{C}$ and $f : U \to V$ a holomorphic bijection with inverse bijection $g : V \to U$. If $f'(a) \neq 0$ for all $a \in U$, then g is holomorphic with derivative*

$$g'(b) \;=\; \frac{1}{f'(g(b))}$$

for all $b \in V$.

Proof. Since $f'(a) \neq 0$ for all $a \in U$, f is locally non-constant. Thus $g^{-1}(W) = f(W)$ is open for every open subset $W \subset U$ by the open mapping principle (Theorem 4.7.1), which shows that g is continuous. Thus the theorem follows from Theorem 1.2.5. □

5.8 Exercises

Exercise 5.8.1. Let $z \in \mathbb{C}$ and $r > 0$ and $f, g : D_r^\bullet(a) \to \mathbb{C}$ be holomorphic and nontrivial. Assume that $\mathrm{ord}_a(f), \mathrm{ord}_a(g) \in \mathbb{Z}$. Show that

$$\begin{aligned} \mathrm{ord}_a(f\cdot g) \;&=\; \mathrm{ord}_a(f)+\mathrm{ord}_a(g); \\ \mathrm{ord}_a(\tfrac{1}{f}) \;&=\; -\mathrm{ord}_a(f); \\ \mathrm{ord}_a(f+g) \;&\geqslant\; \min\{\mathrm{ord}_a(f),\, \mathrm{ord}_a(g)\}. \end{aligned}$$

What happens if $\mathrm{ord}_a(f) = -\infty$?

Exercise 5.8.2. Let $f : D_r^\bullet(a) \to \mathbb{C}$ be holomorphic. Show that a is a removable singularity if and only if f extends analytically to a.

Exercise 5.8.3. Let f be a polynomial of degree d and $a \in \mathbb{C}$. Show that $0 \leqslant \mathrm{ord}_a(f) \leqslant d$.

Exercise 5.8.4. Prove all claims of Example 4.1.3.

Exercise 5.8.5 (Uniqueness of Laurent expansions)**.** Let $f : \mathrm{Ann}_{s,r}(c) \to \mathbb{C}$ be given as

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n\,(z-c)^n$$

where we assume that the Laurent series on the right hand side converges for all $z \in \mathrm{Ann}_{s,r}(c)$. Prove that

$$a_n \;=\; \frac{1}{2\pi\mathbf{i}} \int_{C_t(c)} \frac{f(w)}{(w-c)^{n+1}}\,dw$$

for any $t \in (s,r)$ and conclude that the coefficients a_n are uniquely determined by f (as a function on $\mathrm{Ann}_{s,r}(c)$).

Hint: Replace $f(w)$ by $\sum a_n(w-c)^n$ in the above integral and use that the partial sums converge uniformly to f for $w \in \operatorname{im} C_t(c)$, which allows you to interchange the integral and the infinite sum.

Exercise 5.8.6. Consider the rational function $f(z) = \frac{z}{z^3-1}$.

(1) Show that $\frac{z(z-1)}{z^3-1}$ has a removable singularity at 1. Simplify this expression to find a rational function $g(z)$ that is holomorphic at 1 and such that $f(z) = \frac{1}{z-1} \cdot g(z)$ for all $z \in \mathbb{C} \setminus \{\text{poles of } f\}$.

(2) Determine the terms b_0 and b_1 of the Taylor expansion $g = \sum_{n=0}^{\infty} b_n(z-1)^n$ of g at 1.

(3) Derive the terms a_{-1} and a_0 of the Laurent expansion $f = \sum_{n=-1}^{\infty} a_n(z-1)^n$ of f at 1.

Exercise 5.8.7. Prove the formula of Remark 4.3.4: let $f : D_r^\bullet(c) \to \mathbb{C}$ be holomorphic with Laurent expansion $f(z) = \sum a_n(z-c)^n$ at c and assume that $-m = \mathrm{ord}_c(f) \neq -\infty$. Then for $k \geqslant -m$, we have

$$a_k \;=\; \frac{1}{(m+k)!} \cdot \frac{d^{m+k}}{dz^{m+k}} \Big[(z-c)^m\, f(z)\Big]_{z=c}.$$

Exercise 5.8.8. Prove all claims of Remark 4.4.3.

Exercise 5.8.9 (Geometric intepretation of the winding number)**.** Let $\gamma : I \to \mathbb{C} \setminus \{0\}$ be a closed path and define $\tilde{\gamma} : I \to \mathbb{C} \setminus \{0\}$ by

$$\tilde{\gamma}(t) \;=\; \frac{\gamma(t)}{|\gamma(t)|}.$$

(1) Find a closed homotopy from γ to $\tilde{\gamma}$ in $\mathbb{C} \setminus \{0\}$.

(2) Show that $\tilde{\gamma}(t) = e^{2\pi\mathbf{i}\cdot\alpha(t)}$ for a continuous map $\alpha : I \to \mathbb{R}$ with $\alpha(1) - \alpha(0) \in \mathbb{Z}$.

(3) Let $n = \alpha(1) - \alpha(0)$. Use α to construct a closed homotopy from $\tilde{\gamma}$ to the n-fold circle $\gamma_n(t) = e^{2\pi \mathbf{i} \cdot nt}$.

(4) Conclude that $W(\gamma, 0) = W(\gamma_n, 0) = n$.

Remark: The same arguments work for any point $z_0 \in \mathbb{C} \setminus \operatorname{im} \gamma$. We have assumed $z_0 = 0$ in this exercise merely for simplicity.

Bonus question: Explain why Theorem 4.5.4 is called the argument principle. This is, how does $W(f \circ \gamma, 0)$ relate to the argument of $f \circ \gamma$?

Exercise 5.8.10. Consider the rational function $f(z) = \frac{z}{z^3 - 1}$.

(1) Determine all zeros and poles of f, including their respective orders.

(2) Compute the residue of f at each pole.

(3) Let $\gamma_r = \{1 + re^{it} \mid t \in [0, 2\pi]\}$ for $r = 1, 2$. Determine which poles of f are in its interior for $r = 1, 2$.

(4) Compute

$$\int_{\gamma_r} \frac{z}{z^3 - 1}\, dz$$

for $r = 1, 2$.

Exercise 5.8.11. Prove all claims of Remark 4.5.2.

Exercise 5.8.12. Show that all zeros of the polynomial $f(z) = z^5 + 7z^3 + 10z + 1$ lie in the open disc $D_3(0)$. How many zeros (counted with multiplicities) lie in $D_1(0)$ and in the annuli $\operatorname{Ann}_{1,2}(0)$ and $\operatorname{Ann}_{2,3}(0)$?

Exercise 5.8.13 (Symmetric version of Rouché's theorem)**.** Prove the following stronger version of Rouché's theorem (and think about why it implies Rouché's theorem):

Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ holomorphic and $\gamma : I \to U$ be a closed path that is contractible in U and such that $W(\gamma, z) = 1$ for all $z \in \gamma^{\text{int}}$. If

$$\left|f(\gamma(t)) - g(\gamma(t))\right| \;<\; \left|f(\gamma(t))\right| \;+\; \left|g(\gamma(t))\right|$$

for all $t \in I$, then

$$\sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(f) \quad = \quad \sum_{c \in \gamma^{\text{int}}} \operatorname{ord}_c(g).$$

Hint: You can find an outline of the proof on the Wikipedia page on Rouché's theorem.

Chapter 6

Further topics

6.1 Analytic continuation

Definition 6.1.1. Let $A \subset \mathbb{C}$ be a subset. An *accumulation point of* A is an element $a \in \mathbb{C}$ such that $D^\bullet_\epsilon(a) \cap A \neq \varnothing$ for every $\epsilon > 0$.

Remark 6.1.2. (1) In general, an accumulation point a of A might be in A or not.

(2) The closure $\overline{A}$ of A equals the union of A with all of its accumulation points.

(3) If $\{z_n\}_{n\in\mathbb{N}}$ is a sequence that converges to a, then a is an accumulation point of the set $A = \{z_n\}$. Conversely, if a is an accumulation point of A, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in A that converges to a.

(4) A subset $A \subset \mathbb{C}$ is discrete if and only if it has no accumulation points.

(5) A subset $A \subset \mathbb{C}$ is dense if and only if every element of $\mathbb{C}$ is an accumulation point of A.

Recall that a domain is an open and connected subset of $\mathbb{C}$.

Theorem 6.1.3. *Let U be a domain and $f : U \to \mathbb{C}$ holomorphic and non-trivial. Then $Z = \{z \in \mathbb{C} \mid f(z) = 0\}$ is discrete.*

Proof. Assume that Z is not discrete. Then the set $A \subset U$ of accumulation points of Z in U is non-empty. Consider $a \in A$ and a sequence $\{z_n\}_{n\in\mathbb{N}}$ in Z with limit a. Since f is continuous, we have

$$f(a) \;=\; \lim_{n\to\infty} f(z_n) \;=\; 0,$$

which shows that $a \in Z$. Thus $A \subset Z$, meaning that an accumulation point of A is an accumulation point of Z, and therefore is contained in A. We conclude that $\overline{A} = A$ is closed in U.

Consider $c \in A$ and the Taylor expansion $f(z) = \sum a_n(z-c)^n$ at c (for $z \in D_r(c)$ and suitable $r > 0$ so that $D_r(c) \subset U$). We prove by induction on n that $a_n = 0$ for all $n \in \mathbb{N}$.

The base case $n=0$ follows from the fact that $a_0 = f(c) = 0$ (since $c \in A \subset Z$). Let $n > 0$ and assume that $a_0 = \ldots = a_{n-1} = 0$. Then

$$f_n(z) \;=\; \frac{f(z)}{(z-c)^n} \;=\; \sum_{k=0}^{\infty} a_{n+k}(z-c)^k$$

defines a holomorphic function on $D_r(c)$ with $f_n(z) = 0$ for every $z \in D_r^\bullet(c) \cap Z$. Since c is an accumulation point of Z, and thus of $D_r^\bullet(c) \cap Z$, we conclude as before that $a_n = f_n(c) = 0$, which concludes the induction.

We conclude that $D_r(c) \subset A$, which shows that A is open. Since U is connected and since A is open and closed in U, we conclude that $A = U$. Thus f is constant 0, which contradicts our assumptions. □

Theorem 6.1.4. *Let $U \subset \mathbb{C}$ be a domain $f, g : U \to \mathbb{C}$ holomorphic, and $\{z_n\}_{n\in\mathbb{N}}$ a sequence in U that converges to $a \in U \setminus \{z_n\}_{n\in\mathbb{N}}$. If $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f = g$ as functions.*

Proof. Since $(f-g)(z_n) = 0$ for all $n \in \mathbb{N}$, the set $\{z \in U \mid (f-g)(z) = 0\}$ is not discrete in U since it has an accumulation point. Thus by Theorem 5.1.3, $f - g$ is constant 0, i.e. $f = g$. □

Remark 6.1.5. The hypothesis of Theorem 5.1.4 could be exchanged by either of the following:

(1) $\{z_n\}$ converges to a point $a \in U$ with $\#\{z_n\} = \infty$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$;

(2) $A \subset U$ is a subset with an accumulation point $a \in U$ and $f(z) = g(z)$ for all $z \in A$.

Definition 6.1.6. Let $U \subset \mathbb{C}$ be open, $A \subset U$ a subset and $f : A \to \mathbb{C}$ a function. An *analytic continuation of f to U* is a holomorphic function $\hat{f} : U \to \mathbb{C}$ whose restriction to A equals f.

Note that f extends analytically to U (in the sense of Definition 4.1.1) if and only if it has an analytic continuation to U.

We now prove that analytic continuations are unique.

Corollary 6.1.7. *Let U be a domain, $A \subset U$ a subset with an accumulation point $a \in U$ and $f : A \to \mathbb{C}$ a function. Then f has at most one analytic continuation to U.*

Proof. Since A has an accumulation point a, it contains a sequence $\{z_n\}_{n\in\mathbb{N}}$ that converges to a, yet $z_n \neq a$ for any $n \in \mathbb{N}$. Any two analytic continuations of f to U agree on $\{z_n\}$, as a subset of the domain of f. By Theorem 5.1.4, the two analytic continuations are equal. □

Remark 6.1.8. In the light of Corollary 5.1.7, we can revisit the extensions of real valued analytic functions to holomorphic functions.

Let U be a domain, A a non-empty open subset of $\mathbb{R}$ that is contained in U and $f : A \to \mathbb{R}$ a function; such as $A = \mathbb{R}$ and $U = \mathbb{C}$, or $A = \mathbb{R}_{>0}$ and $U = U_{-1} = \{z \in \mathbb{C} \mid$

$z \notin \mathbb{R}_{\leqslant 0}\}$. As an open subset of $\mathbb{R}$, A has an accumulation point. Thus by Corollary 5.1.7, f has at most one analytic continuation to U.

This re-establishes the holomorphic functions

$$\exp : \mathbb{C} \to \mathbb{C}, \qquad \sin : \mathbb{C} \to \mathbb{C}, \qquad \cos : \mathbb{C} \to \mathbb{C}, \qquad \log : U_{-1} \to \mathbb{C}$$

as the unique analytic continuations of the corresponding real valued functions.

Recipe for analytic continuation. In the following, we explain a strategy for the construction of an analytic continuation of a given holomorphic function $f : V \to \mathbb{C}$ (where $V \subset \mathbb{C}$ is open) to a point $w \in \mathbb{C}$ (typically not in V), by the means of following a path $\gamma : I \to \mathbb{C}$ from a point $\gamma(0) \in V$ to $\gamma(1) = w$. We say that $\hat{f} : U \to \mathbb{C}$ is an *analytic continuation of f along* γ if U contains $V \cup \operatorname{im}\gamma$.

$\boxed{\textbf{Insert an illustration}}$

We aim at extending f iteratively to holomorphic functions $f_i : U_i \to \mathbb{C}$ (for $i \geqslant 0$). As our initial definition (for $i = 0$), we use $t_0 = 0$, $U_0 = V$ and $f_0 = f$.

Let $i > 0$ and assume that we have constructed an analytic continuation $f_{i-1} : U_{i-1} \to \mathbb{C}$ of f to an open superset U_{i-1} of U that contains $\gamma([0, t_{i-1}])$ for some $t_{i-1} \in I$. Since U_{i-1} is open, we can choose a $t_i \in I$ with $t_i > t_{i-1}$ and $\gamma([0, t_i]) \subset U_{i-1}$. Define $c_i = \gamma(t_i)$ and consider the Taylor expansion

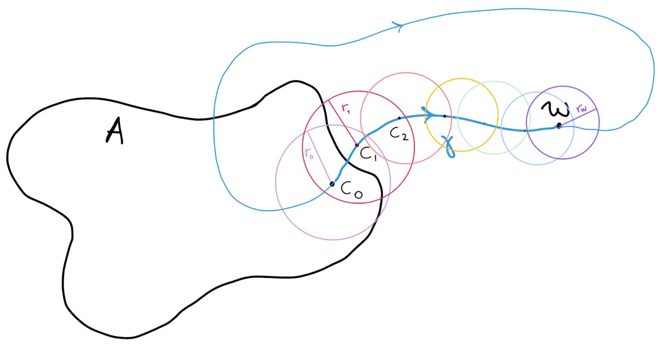
$$f(z) \;=\; \sum_{n=0}^{\infty} a_{i,n}(z - c_i)^n$$

of f at c_i. Let r_i be the radius of convergence of this power series, i.e., $\sum a_{i,n}(z - c_i)^n$ converges for $z \in D_{r_i}(c_i)$ and defines a holomorphic function on this disc, which agrees with f_{i-1} for $z \in D_{r_i}(c_i) \cap U_{i-1}$. We define $U_i = U_{i-1} \cup D_{r_i}(c_i)$ and

$$\begin{array}{rccl} f_i : & U_i & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \begin{cases} f_{i-1}(z) & \text{if } z \in U_{i-1}, \\ \sum a_{i,n}(z - c_i)^n & \text{if } z \in D_{r_i}(c_i), \end{cases} \end{array}$$

which is an analytic continuation of f_{i-1}, and thus of f, to U_i.

This iteration produces a sequence of open subsets $V = U_0 \subset \ldots \subset U_i \subset \ldots$ and analytic continuations $f_i : U_i \to \mathbb{C}$ of f.



Our hope is that eventually $w = \gamma(1) \in U_m$ for some m. In this case, $f_m : U_m \to \mathbb{C}$ is an analytic continuation of f to w along γ.

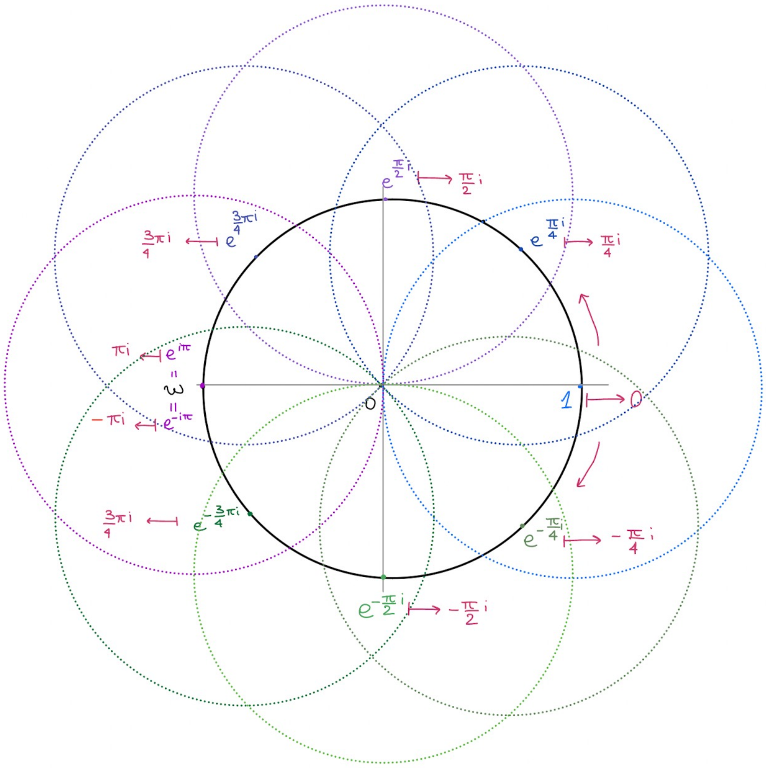
It is, however, not guaranteed that this strategy produces an analytic continuation to w, which depends on f and on a "good" choice of path γ (cf. Exercise 5.2.1). The choices of the t_i are of minor relevance (one merely has to avoid that $\{t_i\}$ forms a converging sequence; cf. Exercise 5.2.2); in particular, the value at w of the analytic continuation of f along γ does not depend on the choices of the t_i. Moreover, the value at $w \in U_m$ does not change if we replace γ by a path $\tilde{\gamma}$ that is homotopic to γ in U_m (rel. to $\{0,1\}$); cf. Exercise 5.2.2.

Note further that the value of $f(w)$ depends in general on the choice of path, provided that our strategy has success (cf. Example 5.1.9).

Example 6.1.9. Consider the restriction $\log : V \to \mathbb{C}$ of the principal branch of the logarithm to $V = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. In the following, we construct analytic continuations to -1, following the half-circles

$$\begin{aligned} \gamma_+ : I &\longrightarrow \mathbb{C} \\ z &\longmapsto e^{\pi \mathbf{i} \cdot t} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \gamma_- : I &\longrightarrow \mathbb{C} \\ z &\longmapsto e^{-\pi \mathbf{i} \cdot t} \end{aligned}$$

through the upper and lower halfplane, respectively, as illustrated as follows:



We see that we can extend log to -1 along both path, but that the value at -1 depends on the chosen path: in the first case, we get $-1 \mapsto \pi\mathbf{i}$, and in the second case, we get $-1 \mapsto -\pi\mathbf{i}$.

6.2 Exercises

Exercise 6.2.1. Consider $f : V \to \mathbb{C}$ with $V = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$ and $f(z) = \frac{1}{z}$ and the path $\gamma : I \to \mathbb{C}$ from 1 to -1 with $\gamma(t) = 1 - 2t$. Show that the Taylor expansion of f at $c = \gamma(t)$ does not converge for $z = 0$ for any $t < 1/2$. Conclude that the strategy for analytic continuation from section 5.1 does not succeed to provide an analytic continuation of f that is defined in -1.

What happens in the case of $\log : V \to \mathbb{C}$ for the same choice of V and γ (cf. Example 5.1.9)? Does log extend to -1 along γ?

Exercise 6.2.2. Let $f : V \to \mathbb{C}$ be a holomorphic function and $\gamma : I \to \mathbb{C}$ a path from $\gamma(0) \in V$ to $w = \gamma(1)$. Let $\hat{f} : U \to \mathbb{C}$ be an analytic continuation along γ.

(1) Show that under these hypothesis, the strategy for analytic continuation along a path from section 5.1 produces an analytic continuation $f_m : U_m \to \mathbb{C}$ of f along γ, and that $f_m(w) = \hat{f}(w)$.

Hint: Show that there is a suitably large n such that we can choose the t_i in such a way such that $t_i - t_{i-1} \geqslant \frac{1}{n}$. Conclude that $w \in U_m$ for some m (in fact, this is the case for some $m \leqslant n$). Use Corollary 5.1.7 to show that $f_m(w) = \hat{f}(w)$.

(2) Let $\tilde{\gamma} : I \to U$ be a path from $\gamma(0)$ to w that is homotopic to γ in U relative to $\{0,1\}$, and $\tilde{f}_{\tilde{m}} : U_{\tilde{m}} \to \mathbb{C}$ an analytic continuation of f to w along $\tilde{\gamma}$. Show that $\tilde{f}_{\tilde{m}}(w) = \hat{f}(w) = f_m(w)$.